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Alternative views on the nature of mathematics and their possible influence on the teaching of mathematics.

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**ALTERNATIVE VIEWS OF THE NATURE OF MATHEMATICS AND THEIR
POSSIBLE INFLUENCE ON THE TEACHING OF MATHEMATICS**

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**Thesis Submitted In Fulfilment Of The
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ABSTRACT

A review of research in mathematics education reveals the lack of adequate theoretical perspectives of mathematics education, and in particular, views of the nature of mathematics. It is suggested that alternative views may significantly affect the teaching of mathematics in distinct ways.

It is proposed, through an examination of schools of thought on the nature of mathematical knowledge, that they can be seen to separate into two streams. There is, firstly, a tendency towards seeing mathematics as based on indubitable, value-free, universal foundations, which may not yet have been completely determined; and secondly a view of mathematics as a social invention, its truths being relative to time and place.

It is further suggested that one can distinguish between two ways of teaching, which reflect this separation, the first being a 'closed' view, whereby the teacher is the possessor of knowledge which is to be conveyed to the recipients, the pupils. The second is concerned with enabling pupils to be actively involved in the processes of doing mathematics, encouraged by 'open' teaching, in the sense of the teacher working from the ideas and concepts of the pupils. These hypothesised positions are not intended to describe an actual teacher, since in practice teachers' views are often not consistent, or even conscious, and their ways of teaching are influenced by other factors also. However, it is maintained that they provide an important theoretical perspective on mathematics education.

A field study is developed to examine some of the consequences of this thesis. A questionnaire is prepared to attempt to identify teachers' views, and an aspect of class teaching proposed as revealing 'open' and 'closed' approaches to mathematics teaching. The study is carried out in one secondary school. From this, a second stage evolves in which the questionnaire is given to a large group of education students, the results analysed, and a sample group of the students interviewed.

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INTRODUCTION

A consideration of theories of mathematics education, purposes, aims, objectives, place in the curriculum, relevance to the real world etc., may best be termed the Philosophy of Mathematics Education. As such, it may be seen as embedded in the Philosophy of Mathematics and the Philosophy of Education. Both, however, are contingent upon one's view of the nature of knowledge, and thus it appears that one must commence such a study here. Problematically, the relationships are in a sense circular:

- (a) Mathematics has traditionally been seen as the paradigm of knowledge, demonstrating certainty, universality, indubitable truth and many other terms with application elsewhere in philosophy. Hence in this sense, knowledge begins with mathematics.
- (b) Any alternative view which brings into question the certainty of mathematical knowledge, would reverse the starting point of consideration.

Education is at least concerned with the transmission of knowledge from society to its students, and hence alternative views of the status of knowledge should have profound effects on education. In particular, I will attempt to show that we in mathematics education tend to direct our ways of teaching, choice of syllabus content etc. on the grounds of the certainty of mathematical knowledge. Hence it may be suggested that we will be most affected by any change in epistemological view.

In Section 1 I will consider the schools of thought on the nature of knowledge in general, and of scientific and mathematical knowledge in particular. I will attempt to show that views on the nature of mathematics can be seen to be either what is termed a 'Euclidean' view (or 'absolutist') or a relativist view (or 'fallibilist'). These views, and some criticisms of each, are discussed and I will attempt to show that fallibility or uncertainty is the more defensible and more

challenging position, demanding imagination and creativity, and endowing mathematics education with excitement and stimulus.

In Section 2 I will consider the connections between theories and the practice of mathematics education. I will attempt to show that fallibilism and absolutism each demand their own particular approach to the teaching of mathematics. It is proposed that two teaching patterns can be identified, which whilst not representing any actual teacher, characterise two ends of a continuum, described as 'open'-'closed', of mathematics teaching behaviour. This section will also consider recent developments in theories of learning mathematics and it is suggested that the constructivist programmes reflect the 'open' end of the continuum and thus also the relativist view of mathematics.

In Section 3 a study is carried out, through two stages, in an attempt to examine some of the implications of the theoretical analysis. A questionnaire is developed, from a group of constructs, through a number of drafts, a pilot test and a validation exercise, to identify teachers' views of mathematics education and mathematics itself, and a marking scheme is developed to assess responses to the questionnaire. An observation tool is adopted, to focus on 'open' and 'closed' teaching, using the criterion of the depth of teacher questions and teacher responses to pupil questions of some depth, if any. The results of the study are discussed, and a second stage study, evolving from this discussion, is developed. This involves having a large group of Postgraduate Certificate of Education students complete the questionnaire, after which some students who scored highest and some who scored lowest on the questionnaire are interviewed individually, after watching an extract of a mathematics lesson, on video. In addition, the questionnaire results of the whole group are analysed, to examine which items are good discriminators, and which are not. These results are then discussed.

Finally, some implications for further study are proposed.

SECTION 1

THE ALTERNATIVES FOR VIEWS OF THE NATURE OF MATHEMATICS

CHAPTER 1 - THE SOCIOLOGY OF KNOWLEDGE

Throughout the history of philosophy, scepticism has always provided a stimulus through its criticism of accepted views. Recent progress in the sociology of knowledge has perhaps provided the strongest sceptical position for criticism of rationality and knowledge, a criticism from which it may be impossible, and indeed unnecessary, to escape.

1.1 The Strong Programme

Much argument centres around the so-called "Strong Programme" in the sociology of knowledge. Its major proponents are based in Edinburgh University, and it has been outlined by David Bloor (1976). He suggests that the sociology of knowledge should adhere to four tenets:

- "1. It would be causal, that is, concerned with the conditions which bring about belief or states of knowledge. Naturally there will be other types of causes apart from social ones which will co-operate in bringing about belief.
2. It would be impartial with respect to truth and falsity, rationality or irrationality, success or failure. Both sides of these dichotomies will require explanation.
3. It would be symmetrical in its style of explanation. The same types of cause would explain, say, true and false beliefs.
4. It would be reflexive. In principle its pattern of explanation would have to be applicable to sociology itself. Like the requirement of symmetry, this is a response to the need to ask for general explanations. It is an obvious requirement of principle because otherwise sociology would be a standing refutation of its own theories." (Page 4)

Perhaps the most controversial of the tenets of the strong programme are the second and the third. The previously accepted view was that true knowledge requires no explanation. According to this view, rationality, correct procedures, clear thinking will inevitably lead to truth, which has a power of its own, by virtue of its own existence. If a scientist

arrives at erroneous conclusions, this is an instance of a clear case for sociological study. But true theories do not need such analysis.

Gilbert Ryle (1949), for example, has written:

"Let the psychologist tell us why we are deceived; but we can tell ourselves and him why we are not deceived." (Page 308)

More recently, Martin Hollis (1982) has said:

"true and rational beliefs need one sort of explanation, false and irrational beliefs another." (Page 75)

David Bloor (1982) suggests, too, that:

"Imre Lakatos was one of the most strident advocates of a structurally similar view. He equated rational procedures in science with those that accord with some preferred philosophy of science. Exhibiting cases which appear to conform to the preferred philosophy is called 'internal history' or 'rational reconstruction'. He then asserts that 'the rational aspect of scientific growth is fully accounted for by one's logic of scientific discovery'. All the rest, which is not fully accounted for, is handed over to the sociologist for non-rational, causal explanation." (page 26)

1.2 Relativism and its Critics

Arising out of anthropological studies, with problems of understanding and interpreting a culture other than that of the observer, and given impetus by Kuhn's work (1970) on scientific cultures, is the relativist position. It is an immediate consequence of the second and third tenets of Bloor's strong programme, that there are no universally acceptable criteria for truth. The justification for conviction of the truth or falsity of a particular topic is dependent on, or relative to, the

context of the individual. In particular, though, as suggested above, the symmetry tenet:

"that all beliefs are on a par with one another with respect to the causes of their credibility." (Barnes 1982, page 23)
is the strongest relativist claim.

One of the major critics of the strong programme is Steven Lukes. In a review of Barry Barnes' book (1974), Lukes summarises Barnes' view as a negative thesis, with which he agrees, and a positive thesis with which he does not agree. He writes (Lukes 1975):

"Barnes has thus far presented a perfectly convincing case against an unwarranted methodological restrictivism, according to which social causation may only be invoked to explain beliefs when they are apparently erroneous or irrational.

The trouble is that throughout this book, Barnes seeks to support his negative thesis by appealing to what I have called his positive thesis. Instead of merely arguing that the apparent truth or rationality of a belief or set of beliefs does not preclude their sociological explanation, he appears to think it necessary to argue that it cannot do so because there are no universally applicable criteria of truth and rationality, and hence beliefs and belief systems cannot be 'explained by a concept of external causes producing deviations from rationality'." (Pages 501-502)

Elsewhere, Lukes (1973) attempts to put a case for universal criteria of truth and rationality, in particular by suggesting that unless one accepts this universality, we cannot discuss, interpret or understand the social activities of another society. The commonality of our interpretations and understandings is just that universality. The issue of incommensurability will be discussed below, but in relation to Lukes' argument, it can be seen that the necessity that he suggests we require is not the case, nor does his argument hold up. As Mary Hesse (1980) has written:

"But even if this were true, it does not show that these criteria are in any sense external or 'absolute', only that they are relative to at least our pair of cultures, rather than to just our culture."
(Page 43)

Hesse goes on to say:

"... this thesis, along with all other epistemologies that reject the possibility of absolute grounds for knowledge ... does not imply that cognitive terminology loses its use, merely that it has to be explicitly redefined to refer to knowledge and truth claims that are relative to some set or sets of cultural norms. These might even be as wide as biological humankind, but if so, they would still not be rendered absolute or transcendentally necessary in themselves. The strong thesis does not imply, however, that there is no distinction between the various kinds of rational rules adopted in a society on the one hand, and their conventions on the other." (Page 56)

Hence the role of epistemologists is not redundant with relativism. Within its set of rational (for that society) rules, there are distinctions between truth and falsity, norms and deviations etc.

"The function of epistemologists to make these things explicit and to study their interrelations is both important and not directly sociological." (Page 46)

Hesse continues with a strong statement of the relativist position, suggesting that those who insist upon a rationalist epistemology are the ones who suffer an emasculation of their theories, not those who accept that criteria of truth and methods of argument are specific to a particular social group, or a number of such groups.

More recently, Lukes has stated his case for universality in, it appears, a much weaker form. He writes (Lukes 1982):

"... what is the significance of the rejection of the traditional folk beliefs in secularizing and modernizing societies, or the seventeenth-century Scientific Revolution? How are such transitions to be interpreted? One answer (though none is definitive) is that a detached, objective and absolute conception of knowledge was in effect isolated and made dominant in certain spheres - even if some of those engaged in the process has a deficient self-understanding of what they were doing." (Page 295)

Whilst we may allow that this is one answer, the apparent at least difficulty, and perhaps impossibility of producing evidence of such absolutes, couple with the renewed interest in folk remedies amongst the most conservative medical profession, and the post-relativity view of the seventeenth-century Scientific Revolution, encourages us to look for a different answer. Indeed Lukes himself ends up proposing a position that he calls 'the soft version of strong perspectivism' which is very close indeed to the relativism he claims to reject.

The crux of the matter appears to be the fear induced amongst philosophers and others by the abandonment of objective criteria for truth, despite Mary Hesse's reassurances. Evidence of this insecurity can be seen, for example, in Louis Boon's critical review of Bloor's book (1976). Boon (1979) clearly states that he found the book hard to read, since:

"Bloor seems to rely on the strategic principles of the B-movie: the baddies (in this case the philosophers) are dumb." (Page 195)

He proceeds to attempt to demolish the whole programme for the sociology of knowledge, suggesting that ultimately one must conclude from Bloor's arguments that:

"naturalism leads to a form of the cunning of reason as the agent of progress in knowledge." (Page 195)

He manages however to avoid a serious consideration of Bloor's book.

A.F.Chalmers, in a critical review of a book by Harold Brown (1977), recognises the relativist position being adopted (Chalmers 1979(a)):

"The author insists that criteria of adequacy are internal to a science, change in time, and must be evaluated with respect to the theoretical and historical situation." (Page 97)

Later, however, whilst accepting that Brown's case is a strong one, Chalmers feels the need to supplement Brown's argument, by putting forward:

"an objectivist non-relativist account of science which construes theory change, not in terms of the decisions made by individual scientists or groups of scientists, but in terms of the objective properties of theories..." (Page 97)

Chalmers exhibits here this apparent need shown by many scientists and philosophers for some objectivity, somewhere, on which to hold. His account of theory change in objective terms is one possibility. We have already seen Steven Lukes' attempt, which in the end he himself appears to have abandoned. Chalmers lays out his case for theory change in an attempt to strengthen Lakatos' methodology of scientific research programmes, which Chalmers considers is vulnerable to Feyerabend's criticism of anarchy, namely, that decisions of adoption of alternative research programmes do not, according to Feyerabend, fit any rational pattern (e.g. Feyerabend 1978). Chalmers (1979(b)) maintains that:

"Theory change can be understood as coming about by virtue of the fact that an established theory was challenged and ousted by a rival that offered more objective opportunities for development, some at least of which bore fruit. This contrasts with attempts to explain theory change by reference to the rationality or otherwise of the decisions and choices of individual scientists." (Page 231)

He recognises a constraint, however, and admits that:

"... the fact that a programme presents opportunities for development is no guarantee that those opportunities will lead to actual success when taken up. Whether or not a research programme supersedes a rival will depend, not solely on its degree of fertility, but also on its success in practice." (Page 231)

Chalmers' second point reveals the weakness of his argument. Even looking backwards at stages where such choices had to be made, it is at least problematic to exactly determine which programme offered more objective opportunities for development, since, by his own admission, the successful development of these opportunities depends on all sorts of other factors. An assessment of whether one theory has actually superseded another would have to be made relative to the outcome, which was dependent on all these various factors. It certainly seems to be the case that at the stage of such a choice, the relative merits of alternative programmes could hardly be objectively determined. Such an assessment would be made in the light of the prejudices of the scientific community at the time, the position, social and cognitive, of the individual scientist or group of scientists, and many other factors. In any case, it does not seem to make the study of the history of science or the philosophy of science less relevant if one accepts the existential nature of the state of knowledge at any given time.

Martin Hollis, in an article entitled "The Social Destruction of Reality" (1982) claims that the relativist programme leads to what he calls 'a lethal dry rot', in that since epistemology and ontology are both relative, subject to an overall coherence, and since the terms of that coherence are also relative, being included in the epistemology, there is no constraint left. He suggests and then rejects the possibility of stopping the rot by accepting an objective world argument, and instead maintains firstly a 'bridgehead' of concepts shared by all cultures to avoid incommensurability, and:

"The other way, then, is to place an a priori constraint on what a rational man can believe about his world. On transcendentalist grounds there has to be that 'massive central core of human thinking

which has no history' and it has to be one which embodies the only kind of rational thinking there can be. The 'massive central core' cannot be an empirical hypothesis, liable in principle to be falsified in the variety of human cultures but luckily in fact upheld... the existence of a core must be taken as a precondition of understanding beliefs. There has to be an epistemological unit of mankind.

The plain snag here is that such reflections yield at most an existence proof." (Page 83)

The criticism that relativism has no constraints is a common one amongst those who are seeking to justify the existence of absolutes. It is an important criticism, and must be answered, and Wittgenstein provides an answer.

"The procedure of putting a lump of cheese on a balance and fixing the price by the turn of the scale would lose its point, if it frequently happened for such lumps to suddenly grow or shrink for no apparent reason." (Bloor 1973 page 184)

"Strange coincidence, that every man whose skull has been opened had a brain!" (Wittgenstein 1979 para 207)

"I have a telephone conversation with New York. My friend tells me that his young trees have buds of such and such kind. I am now convinced that his tree is... Am I also convinced that the earth exists?

... The existence of the earth is rather part of the whole picture which forms the starting-point of belief for me." (Wittgenstein 1969 paras 207, 209)

It is not necessary to appeal to objective knowledge, a priori knowledge or absolute certainty. Wittgenstein is here showing that whilst in

principle, logically, we can invent any fictitious natural history, in practice we are constrained by facts of nature.

"If humans were not in general agreed about the colour of things, if undetermined cases were not exceptional, then our concepts of colour could not exist! No:- our concept would not exist." (Wittgenstein 1967(a) para 351)

"Do I want to say, then, that certain facts are favourable to the formation of certain concepts; or again unfavourable? And does experience teach us this? It is a fact of experience that human beings alter their concepts, exchange them for others when they learn new facts; when in this way what was formerly important to them becomes unimportant, and vice versa." (Wittgenstein 1967(a) para 352)

It is perfectly adequate to proceed as scientists, philosophers, epistemologists and others from 'facts of nature' and not to have to demand universals. To repeat the point made above by Mary Hesse, there is plenty for us to do, within our perspective, sorting out correctness and error, truth and falsity and so on. These occupations are as vital when endowed with relativist values and perhaps more so.

This point about the use of terms like true and false by a relativist is highlighted by David Bloor, in a reply to a criticism by Steven Lukes on Bloor's article "Durkheim and Mauss Revisited: Classification and the Sociology of Knowledge" (Bloor 1982):

"Another objection concerns my use of the words 'true' and 'false'. Lukes says I have no right to use those terms, given the relativist position that I am developing in this paper...

First, when a relativist is describing the beliefs of, say, the corpuscular philosophers, he may have occasion to say what they designated as true and false. Similarly, when addressing an argument to readers who cannot themselves be assumed to be

relativists, then the terms represent a convenient shorthand. The natural way to recommend say, a relativist methodology would be to suggest that both true and false beliefs should be treated as equally problematic by the sociologist of knowledge... Finally, despite what Lukes supposes, even from the relativist standpoint itself, there is no reason for totally discarding words like 'true' and 'false'. There is a simple relativist analysis of what is involved in their use: these terms simply signalize some important practical discriminations. They are an idiom of acceptance and rejection. Everyone needs to treat beliefs and claims selectively in the conduct of their practical and theoretical affairs. What the relativist says is that the justifications that can be given for these selections (including his own) will be relative to time and place and of merely local credibility. Bereft of metaphysical pretensions, the words 'true' and 'false' still retain their immediate, local and practical import. Are believers in a flat earth the only ones amongst us with the right to operate with the distinction between 'up' and 'down'?" (Page 321)

1.3 Incommensurability

As mentioned above, incommensurability of different cultures or communities is a serious criticism of relativist theories. Kuhn, in his book "The Structure of Scientific Revolutions" (1970) claims that to be within a scientific community is to hold the paradigm of that community. Any paradigm shift that occurs, that is the conversion from one paradigm to another, has to be like a Gestalt switch in that it must take place in a flash. Hence, Kuhn maintains, it is impossible to hold two competing paradigms at the same time. He says:

"Therefore, at times of revolution, when the normal-scientific tradition changes, the scientists's perception of his environment must be re-educated - in some familiar situations he must learn to see a new gestalt. After he has done so the world of his research will seem, here and there, incommensurable with the one he had inhabited before." (Page 11)

Feyerabend (1978) supports this view of the incommensurability of rival theories. He writes, for example:

"Incommensurable theories, then, can be refuted by reference to their own respective kinds of experience; i.e. by discovering the internal contradictions from which they are suffering. (In the absence of commensurable alternatives these refutations are quite weak, however,...) Their contents cannot be compared. Nor is it possible to make a judgement of verisimilitude except within the confines of a particular theory (remember that the problem of incommensurability arises only when we analyse the change of comprehensive cosmological points of view - restricted theories rarely lead to the needed conceptual revisions)." (Page 284)

Feyerabend is of course presenting here his own brand of philosophy of anarchism, in which there are no criteria for preferring any alternative theory.

It is interesting to note that Kuhn and Feyerabend are both counter-examples to their own theories of the impossibility of seeing two rival theories at the same time. Kuhn (1970) writes, for example:

"How am I to persuade Sir Karl, who knows everything I know about scientific development and who has somewhere or other said it, that what he calls a duck can be seen as a rabbit? How am I to show him what it would be like to wear my spectacles when he has already learned to look at everything I can point to through his own?" (Page 3)

One possible counter to incommensurability, as we have seen above, is to suggest that as it must be possible to look at other societies and cultures and understand what is going on, and in science to understand two rival theories at the same time, then there must be at least a bridgehead of concepts that are of necessity in common to all cultures. However, again as we have seen above, it seems impossible to determine any incontrovertible content to this bridgehead of concepts.

There are, it seems, at least two ways in which the problems of incommensurability can be seen to disappear: there may be a fundamental misunderstanding of a gestalt switch; there is no necessary bridgehead, but there are facts of nature, in the Wittgensteinian sense.

1.3.1 There may be a fundamental misunderstanding of a gestalt switch

Kuhn, Feyerabend and others often use diagrams of objects that can be seen, at different times by the same person, or by different people at the same time, to be two different objects, e.g. Kohler's goblet and faces drawing. They claim that what is happening here is that it is impossible to see both objects at the same time. In an analogous fashion, one cannot see two alternative world views at the same time. However, it is not the case that one says "I can only see the goblets", or "I can only see the two faces". One says, "At this instant I can see one, whereas just after I can see the other drawing". One is, in a sense straddling both images, or paradigms, at the same time, able to see both while at any moment holding one or the other in view. This certainly fits Wittgenstein's description of different language games that overlap, like intersecting sets. As mentioned above, Kuhn is just such an example, of a person who is able to see competing theories, which he would presumably see as incommensurable, the Popperian world and the Kuhnian world, at the same time.

Derek Phillips uses an example of a person who at school may see an object as a glass and metal instrument, and after training would see it as an X-ray tube, with all the knowledge which is associated with understanding the working and functions of such a machine. Again, it is not the case that the person would only see the object in one way, unable to bring the two images together. The person would be more likely to say "Before I saw that as a glass and metal object, and I can see how I only saw it that way, but now I recognise it to be an X-ray tube." (Phillips 1977 page 104). Again this hypothetical person is straddling two rival views at the same time.

To take this argument into the heart of Kuhn's concern, scientific revolutions, again it seems more reasonable to say, that whilst the insight of a new rival paradigm may be instantaneous, in a flash as it were, the 'training' leading up to the gestalt switch, to use Kuhn's own image, would have been a gradual process of doubts, inconsistencies, rival ideas read or heard. After the switch, as with our two examples above, the scientist would be more likely to say, "I can see how I used to think that, but now I see it this way", rather than to suddenly find himself/herself unable to communicate with colleagues who, moments before were in the same scientific community.

1.3.2 Wittgensteinian 'facts of nature'

Whilst we can imagine alternative cultures, or world views, even ones that would clearly have great difficulty understanding each other's concepts, there are still underlying facts of nature. One cannot ascribe necessity or absoluteness to them, but they are nevertheless facts of nature of our common world. Wittgenstein (1967(b)) writes, for example:

"'There are 60 seconds to a minute.' This proposition is very like a mathematical one. Does its truth depend on experience? - Well, could we talk about minutes and hours, if we had no sense of time; if there were no clocks, or could be none for physical reasons; if there did not exist all the connexions that give our measures of time meaning and importance?" (Section V para 15)

or in another case

"What we are supplying are really remarks of the natural history of men: not curiosities, however, but rather, observations on facts which noone has doubted and which have only gone unremarked because they are always before our eyes" (Section 1 para 141)

and finally

"The limitations of empiricism are not assumptions unguaranteed, or intuitively known to be correct: they are ways in which we make comparisons and in which we act." (Section V para 18)

Wittgenstein himself illustrates the commensurability of alternative rival theories. His early work, the *Tractatus*, and his later work *Philosophical Investigations* are opposing views of knowledge. The later work is written as a dialogue between the early Wittgenstein and the later one, as the first extract above illustrates. There is no difficulty in this for Wittgenstein. He understands his former position, and is in dialogue with himself to present his later views.

1.4 Summary

Scientific philosophy today has here been characterised as an ongoing and somewhat heated debate between the proponents of relativism and those wishing to provide some secure and objective basis to knowledge in general and scientific knowledge in particular. It appears that the motive of the opponents of relativism is the fear that we have no firm foundations, no certainty, without some way of judging progress, if not truth itself, with universal objective criteria. On one side, any attempt to identify universals seems to fail in the light of relativistic arguments. On the other side, Kuhn appears to wish to draw back from the edge of irrationalism, although his arguments do not allow him to do so, whilst Feyerabend has no hesitation in stepping over that borderline. Common sense suggests that there is such a thing as progress, certainly over a period of time. This is inadequate in a search for universal criteria, but perfectly adequate from the Wittgensteinian position suggested here. In any case, it has been suggested here that the fears of the absolutists are unnecessary. Indeed Mary Hesse encourages scientists with the thought that we are better off working from a relativist position.

1.5 Sociology of Mathematics

If there is a strong prejudice amongst scientists against a strong position in the sociology of knowledge, it must surely be stronger still amongst mathematicians, since we are inclined to consider mathematical concepts as somehow a priori, even if there are no others than in mathematics. In the next chapter we will consider the state of the philosophy of mathematics, but in this section I propose to examine the current literature in the sociology of mathematics. This will be followed by a discussion of the role of mathematics education in social control within schools.

Bloor (1976) outlines the problems facing a sociological analysis of mathematics knowledge:

"Everyone accepts that it is possible to have a relatively modest sociology of mathematics studying professional recruitment, career patterns and similar topics. This might justly be called the sociology of mathematicians rather than of mathematics. A more controversial question is whether sociology can touch the very heart of mathematical knowledge. Can it explain the logical necessity of a step in reasoning or why a proof is in fact a proof? The best answer to these questions is to provide examples of such sociological analyses, and I shall attempt to do this. It must be admitted that these 'constructive' proofs cannot be offered in abundance. The reason is that mathematics is typically thought about in ways which obscure the possibility of such investigations. An enormous amount of work is devoted to maintaining a perspective which forbids a sociological standpoint. By exhibiting the tactics that are adopted to achieve this end, I hope to convey the idea that there is nothing obvious, natural or compelling about seeing mathematics as a special case which will forever defy the scrutiny of the social scientist. Indeed I shall show that the opposite is the case. To see mathematics as surrounded by a protective aura is often a strained, difficult and anxiety-ridden stance. Furthermore

it leads its advocates to adopt positions at variance with the accepted spirit of scientific inquiry." (Page 74)

As Bloor has said above, if sociological accounts can be produced of the choice of alternative theories by mathematicians, and even alternative conceptions of the nature of mathematics, and of mathematical truth, then Bloor's argument has indeed strong support. He proceeds to attempt to do this in his book, and there have been a number of sociological studies published since, in the same vein.

Joan L. Richards (1979) has investigated the attitudes of British mathematicians to non-Euclidean geometry in the 19th century:

"In order to understand the kinds of implications that were attendant on non-Euclidean geometry in the 1870's, when Riemann's and Helmholtz's ideas were introduced, it is first necessary to outline the position which geometry held within English philosophical traditions. When this position is clear, the implications which were inherent in non-Euclidean geometry will be easier to understand. A discussion of philosophical tradition will shed light on the conflicts which were developing in the 1860's and 1870's over the status of scientific knowledge. It was in the midst of this controversy that the new geometrical ideas were introduced, and they had important implications for these discussions. Within this context, the reaction of English mathematicians as a group, to geometrical developments, their tendency to develop geometry within the projective framework rather than the differential one, makes a great deal of sense. In large part it represents a conservative reaction against the new geometry, an attempt to maintain the status quo against the broad impact of differential geometry." (Page 145)

In another article, this time dealing with algebra, Joan Richards (1980) attempts to show that the British conception of truth in particular in mathematics, meant that British mathematicians in the 19th century failed to develop abstract algebra. They had, she maintains, recognised a formalist view of mathematical development, but for them it was

meaningless in the view of truth which they saw exemplified by mathematics.

David Bloor presented a paper entitled "Did Hamilton's metaphysics influence his algebra?" at a workshop on the Social History of Mathematics in 1979. In this paper, he criticizes the work of Thomas Hawkins (1976, 1977) who attempted to show that Hamilton's work, in particular his development of quaternions, was a result of his metaphysics. Bloor suggests firstly that a case can equally be put that Hamilton only couched his results in terms which fit his idealist view of mathematics, but that his work is clearly in the intuitive, problem-solving tradition; secondly he attempts to show that the way one construes reality is involved with social control, the dominance of one group over another. He writes:

"I think that the role of Hamilton's metaphysics is best understood by examining his attitude to the 'formalism' of the Cambridge school. It is well known that Hamilton was hostile to formalism. He said that if we abandon the idea of an independent truth for mathematics then the "Symbols will become what many now account them to be, the all-in-all of algebra". He said that his reaction to Peacock's Treatise was that it would "reduce algebra to a mere system of symbols, and nothing more, an affair of pothooks and hangers, of black strokes on white paper."

The question that I think should be asked is: What is happening when some mathematicians treat symbols as if they were self-sufficient things and see mathematics as marks on paper, whilst others demand that symbols have a reference and meaning that makes them more than mere marks?

My answer is that attitudes towards symbols are themselves symbolic. I suggest that man will impute self-sufficiency to their symbols when they, their users, are asserting their own self-sufficiency or impressing their independence on others. Conversely symbols will be portrayed as standing in need of reference to something ideal when

their users want to impress on others the need for an analogous dependence in the social realm. To be a formalist is to say: "We can take charge of our own destiny". To reject formalism is to reject this message. It is therefore an appropriate way of endorsing the established institutions of social control, especially the traditional means of spiritual guidance." (Page 12)

This is a similar kind of analysis of alternative theories as that shown by Bloor in his "Knowledge and Social Imagery" (1976). Here, in considering the Popper/Kuhn debate, he describes Enlightenment and Romantic ideologies, places Popper in the former and Kuhn in the latter, and characterises their conflict as a typical clash between these two ideologies and their rival social theories. What is significant for us here is not whether Bloor is correct, or that Bloor's analysis of the Popper/Kuhn debate actually explains more than another sociological analysis, but that sociological analyses are actually taking place. Scientific method, the comparison of rival theories for their greater or lesser explanatory power, is being applied to scientific knowledge itself, with some considerable success, and quite independent of the truth or falsity of the mathematical theory. We shall return to the issue implied in the last paragraph quoted from Bloor's paper, and in particular the last sentence, spiritual guidance, in the next section, with reference to education.

Judith Grabiner, in an article entitled "Is mathematical truth time-dependent?" (1974) looks at the work of Euler, Cauchy and Weierstrass on infinitesimals and rigour. She concludes:

"Perhaps mathematical truth is eternal but our knowledge of it is not. We have seen an example of how attitudes towards mathematical truth have changed in time. After such a revolution in thought, earlier work is re-evaluated. Some is considered worth more; some worth less.

What should a mathematician do knowing such re-evaluations occur?

... I suggest a third possibility: a recognition that the problem I have raised is just the existential situation mathematicians find themselves in. Mathematics grows in two ways: not only by successive increments, but also by occasional revolutions. Only if we accept the possibility of present error, can we hope that the future will bring a fundamental improvement in our knowledge. We can be consoled that most of the old bricks will find places somewhere in the new structure. Mathematics is not the unique science without revolutions. Rather mathematics is that area of human activity which has at once the least destructive and still the most fundamental revolutions." (Page 364)

Bos and Mehrtens (1977) have attempted to give some structure to the sociology of mathematics, in an article entitled "The Interactions of Mathematics and Society in History - Some Explanatory Remarks." They write:

"In accordance with its special purpose, the paper has three aims. First it argues for the importance of the subject, which we feel, deserves more attention from historians of mathematics than it receives at the moment. In part 1 we discuss three arguments for studying the relation of mathematics and society and for treating this theme in teaching. The second aim, treated in part 2, is to provide a preliminary exploration of the roles which mathematics may play in society. These concepts are, of course, debatable, but we hope they may be helpful in a further discussion. The third aim, the subject of part 3, is to mention a number of specific themes on which research on the relation of mathematics and society could focus, and to provide references to literature, in particular for a discussion of these themes in teaching." (Page 7)

The quantity of literature on the sociology of mathematics, while remaining far behind the literature on the sociology of science, is nevertheless growing, and regular workshops are being held in the social history of mathematics. One can expect much research to emerge,

providing the examples that Bloor suggests will demonstrate the strong programme.

1.6 Mathematics Education and Social Control

I propose here, taking Bloor's analysis, to show that the two schools of thought in the philosophy of mathematics discussed in chapter 2 of this thesis, namely the Euclidean programme and the Lakatosian alternative programme, and the two ways of teaching discussed in chapter 4 of this thesis, namely 'closed' and 'open', can be seen as rival conceptions of the aims and purposes of education. Teaching mathematics as a body of knowledge can be characterised as one particular view of the relationship between teacher and pupil, that of the learned and the learner, the possessor of knowledge and the receiver of knowledge, the controller and the controlled. Teaching mathematics as a way of thinking, on the other hand, can be seen, with its dynamic set of methods, techniques and development of intuitive skills, as another view, that of encouraging the creative process that each individual learner goes through in the process called learning, and that is the role of the teacher to the pupil. The latter conception can be called child-centred, in that the emphasis is on the creative process that the pupil must go through for learning to take place. A shift away from the metaphysical status of mathematical knowledge, and towards the patterns of thought and behaviour that identify mathematics, is also a shift away from the control of one group by another by virtue of its privileged position in relation to knowledge, and towards a form of control more by acceptance of one group of the greater experience of the other. As Bloor (1979) has described it:

"We should start with the idea that in our social interactions we are always trying to put pressure on our fellows or evade that pressure. The crucial point is that in order to apply pressure more effectively we try to make reality our ally. We construe reality in such a way that it justifies or legitimates our course of action."

(Page 13)

We can certainly see mathematics education in this light. In earlier days mathematics was seen to be the paradigm of certain knowledge, and consequently the model for systems of moral knowledge, scientific knowledge etc., particularly in the Euclidean style. It is probably the case that to a large extent mathematics is still seen in this light by many teachers of mathematics, although not consciously. This will be considered further in Section 3 of this thesis. The ability to appeal to a higher authority for certainty is, in the traditional mode, a necessary tool for social control. Deviant behaviour is then clearly identified and can be excluded.

An interesting analogy has been drawn by Bloor (1978) between monster-barring techniques in mathematical development, as outlined so clearly in Lakatos' book "Proofs and Refutations" (1977), and the exclusion of animals which do not fit a specific categorisation in Jewish Dietary Laws, described by Mary Douglas in her book "Natural Symbols: Explorations in Cosmology" (1973). Bloor writes:

"These books have a common theme: they deal with the way man responds to things which do not fit into the boxes and boundaries of accepted ways of thinking: they are about anomalies to publicly-accepted schemes of classification. Whether it be a counter-example to a proof; an animal which does not fit into the local taxonomy; or a deviant who violates the current norms, the same range of reactions is generated....

The crucial point is that Mary Douglas has an explanation of why there are different responses to things which break the orderly boundaries of our thinking: these responses are characteristic of different social structures. Her theory spells out why this will be so, and describes some of the mechanisms linking the social and the cognitive. This means we should be able to predict the social circumstances which lie behind the different responses which mathematicians make to the troubles in their proofs." (Page 245)

Similarly, we may see the responses which mathematics teachers make to the troubles in their classrooms. We perpetuate the view that mathematics is an esoteric affair. We have the knowledge, both of the correct way of doing any specific piece of mathematics, and of the significance and relevance of a particular piece of work. We do not really need to explain to an inquisitive pupil. We can merely state that it is too complicated to explain, the word of the teacher will have to be accepted; or it is on the examination syllabus, and that higher authority, the examiner, is also quite adequate justification. Of all the subjects in the school curriculum, mathematics appeals most as an authority-based subject. This probably explains why a mathematics exercise is most often given as a disciplinary measure. Mathematical behaviour is right or wrong, and the higher authority determines which. It is a clear analogy with moral behaviour.

Alternatively, pupils can receive the view that they actively participate in the learning process, and that without their activity, their learning does not take place. Mathematics can be mastered by all, to some degree, since it is a way of dealing with a certain set of experiences encountered in interaction with the world around. Teachers are seen as those with more experience of mathematizing, who can usually lead pupils in the best direction to explore and develop those perceptions.

It can often happen that pupils have a perception of a particular problem that is quite novel, whether correct or not. In fact at first the teacher may not be sure whether the approach is correct. If the teacher acknowledges the pupil's response, encourages the pupil and the class to examine the idea, test it, generalise it etc., the teacher may be seen to be demonstrating the notion that mathematical knowledge is not the exclusive domain of the teacher, providing the teacher with a position of authority in the social interactions as well as the knowledge.

It may be, then, that an epistemological commitment is also a commitment to a form of social interaction in the classroom, providing

the teacher with the authority of the possessor of knowledge, or as the guide and adviser to pupils in the learning process.

1.7 Sociology of Mathematics - A Summary

From the perspective of relativism, studies of mathematics discussed provide exciting new pictures of the nature of the development of mathematical knowledge, symmetrical with respect of truth and falsity, and reflexive with respect to sociological studies themselves. The commitments in a social sense that result from epistemological positions adopted are revealed. The social nature of mathematics is convincingly argued for. Derek Phillips (1977) writes, giving a Wittgensteinian image:

"... measuring, calculating, inferring and so forth, are bounded by facts of nature, but particular systems of measurement, calculation and so on, are fully a matter of social convention. One or another type of mathematics is invented or created against the background of a certain consistency of objects in nature (they do not suddenly change size or shape, they do not suddenly disappear), the human capacity to remember numbers accurately, and the like, the various uses that counting and calculating have in our lives and so forth. But while these facts of nature set certain limitations as to the possibilities of various language games - including mathematics - they can account neither for the existence of particular language games nor for the manner in which people learn to play those games."

(Page 135)

1.8 Conclusion

As has been suggested above, a common tactic amongst scientists and philosophers has been to fall back on mathematics as the form of knowledge with certainty built into it. As Lakatos (1978) has described it:

"Classical epistemology has for two thousand years modelled its ideal of a theory, whether scientific or mathematical, on its conception of Euclidean geometry." (Page 29)

"By the turn of this century mathematics, 'the paradigm of certainty and truth', seemed to be the last stronghold of orthodox Euclideanism." (Page 30)

In Chapter 2 the situation in the philosophy of mathematics will be discussed.

CHAPTER 2 - THE PHILOSOPHY OF MATHEMATICS

A brief summary will first be given of the three traditional schools of thought in the philosophy of mathematics, namely logicism, formalism and intuitionism. This will be followed by an examination of the theories of Imre Lakatos in relation to the nature of mathematical knowledge, and other recent work along the same lines. In particular, the effects of the loss of the traditional certainty of mathematical knowledge will be discussed, and implications for mathematics education suggested.

2.1 Logicism

Carl Hempel has described the thesis of logicism concerning the nature of mathematics in the following way (Benacerraf 1964):

"Mathematics is a branch of logic. It can be derived from logic in the following sense:

- a. All the concepts of mathematics, i.e. of arithmetic, algebra, and analysis, can be defined in terms of four concepts of pure logic.
- b. All the theorems of mathematics can be deduced from those definitions by means of the principles of logic (including the axioms of infinity and choice).

In this sense it can be said that the propositions of the system of mathematics as here delimited are true by virtue of the definitions of the mathematical concepts involved, or that they make explicit certain characteristics with which we have endowed our mathematical concepts by definition. The propositions of mathematics have, therefore, the same unquestionable certainty which is typical of such propositions as "All bachelors are unmarried", but they also share the complete lack of empirical content which is associated with that certainty: The propositions of mathematics are devoid of all factual content: they convey no information whatever on any empirical subject matter." (Page 378)

Hempel goes on to discuss the apparent paradox in that despite this emptiness of factual content mathematics applies to empirical subject matter. He explains this as follows:

"Thus, in the establishment of empirical knowledge, mathematics (as well as logic) has, so to speak, the function of a theoretical juice extractor: the techniques of mathematical and logical theory can produce no more juice of factual information than is contained in the assumptions to which they are applied; but they may produce a great deal more juice of this kind than might have been anticipated upon a first intuitive inspection of those assumptions which form the raw material for the extractor." (Page 379)

Even before Godel's incompleteness results, there were fundamental problems with the logicist programme. As Russell wrote in his "Introduction to Mathematical Philosophy" (Benacerraf 1964):

"... But although all logical (or mathematical) propositions can be expressed wholly in terms of logical constants together with variables, it is not the case that, conversely, all propositions that can be expressed in this way are logical. We have found so far a necessary but not a sufficient criterion of mathematical propositions. We have sufficiently defined the character of the primitive ideas in terms of which all the ideas of mathematics can be defined, but not of the primitive propositions from which all the propositions of mathematics can be deduced. This is a more difficult matter, as to which it is not yet known what the full answer is.

We may take the axiom of infinity as an example of a proposition which, although it can be enunciated in logical terms cannot be asserted by logic to be true." (Page 130)

Carnap has written (Benacerraf 1964):

"A greater difficulty, perhaps the greatest difficulty, in the construction of mathematics has to do with another axiom posited by Russell, the so-called axiom of reducibility, which has justly become the main bone of contention for the critics of the system of Principia Mathematica. We agree with the opponents of logicism that it is unadmissible to take it as an axiom." (Page 35)

Godel's work appears to have put paid to the logicist programme entirely, in showing that, given any consistent set of arithmetical axioms, there are true arithmetical statements which are not derivable from the set.

2.2 Formalism

Hilbert's Formalist programme has been summarised by Lakatos (1978):

"How could we test Russellian logic? All true basic statements - the decidable kernel of arithmetic and logic - are derivable in it, and thus does not seem to have any potential falsifiers. So the only way of criticising this peculiar empiricist theory is, on the face of it, to test it for consistency. This leads us to the Hilbertian circle of ideas.

Hilbertian meta-mathematics was 'designed to put an end to scepticism once and for all'. Thus its aim was identical with that of the logicians.

"One has to admit that in the long run the situation in which we find ourselves because of the paradoxes is an unbearable one. Just imagine: in mathematics, in this paradigm of certainty and truth, the most common concept-formations and inferences that are learned, taught and used, lead to absurdities. But if even mathematics fails, where are we to look for certainty and truth? There is however a completely satisfactory method of avoiding paradoxes."

Hilbert's theory was based on the idea of formal axiomatics. He claimed (a) that all arithmetical propositions which are formally proved - the arithmetical axioms - will certainly be true if the

formal system is consistent, in the sense that A and not-A are not both theorems, (b) that all mathematical truths can be proved, and (c) that meta-mathematics, this new branch of mathematics set up to prove the consistency and completeness of formal systems, will be a particular brand of Euclidean theory: a 'finitary' theory, with trivially true axioms containing only perfectly well known terms, and with trivially safe inferences. 'It is contended that the principles used in the meta-mathematical proof that the axioms do not lead to contradiction, are so obviously true that not even the sceptics can doubt them'. A meta-mathematical argument will be 'a concatenation of self-evident intuitive (inhaltlich) insights'. Arithmetical truth - and, because of the already accomplished arithmetization of mathematics, all sorts of mathematical truths - will rest on a firm, trivial, 'global' intuition, and thus on 'absolute truth'." (Page 20)

Von Neumann has written (Benacerraf 1964):

"The leading idea of Hilbert's theory of proof is that, even if the statements of classical mathematics should turn out to be false as to content, nevertheless, classical mathematics involves an internally closed procedure which operates according to fixed rules known to all mathematicians and which consists basically in constructing successively certain combinations of primitive symbols which are considered 'correct' or 'proved'. This construction-procedure, moreover, is 'finitary' and directly constructive... although the content of a classical mathematical sentence cannot always (i.e. generally) be finitely verified, the formal way in which we arrive at the sentence can be... We must regard classical mathematics as a combinatorial game played with primitive symbols..." (Page 50)

One can see here how devastating Godel's proof was for both formalism and logicism together. A formalist approach to the nature of mathematics is, however, far from dead, as will be discussed below.

2.3 Intuitionism

The two main exponents of intuitionism are Arend Heyting and L.E.J.Brouwer. Heyting, firstly, writes (Benacerraf 1964):

"The intuitionist mathematician proposes to do mathematics as a natural function of his intellect, as a free, vital activity of thought. For him, mathematics is a production of the human mind. He uses language, both natural and formalized, only for communicating thoughts, i.e., to get others or himself to follow his own mathematical ideas. Such a linguistic accompaniment is not a representation of mathematics; still less is it mathematics itself... I must... make one remark which is essential for a correct understanding of our intuitionist position: we do not attribute an existence independent of our thought, i.e. a transcendental existence, to the integers or any other mathematical objects. Even though it might be true that every thought refers to an object conceived to exist independently of it, we can nevertheless let this remain an open question. In any event, such an object need not be completely independent of human thought. Even if they should be independent of individual acts of thought, mathematical objects are by their very nature dependent on human thought. Their existence is guaranteed only in so far as they can be determined by thought. They have properties only in so far as these can be discerned in them by thought. But this possibility of knowledge is revealed to us only by the act of knowing itself. Faith in transcendental existence, unsupported by concepts, must be rejected as a means of mathematical proof. As I will shortly illustrate more fully by an example, this is the reason for doubting the law of the excluded middle." (Page 43)

Brouwer writes (Benacerraf 1964):

"The... point of view that there are no non-experienced truths and that logic is not an absolutely reliable instrument to discover truths, has found acceptance with regard to mathematics much later

than with regard to practical life and science. Mathematics rigorously treated from this point of view, and deducing theorems exclusively by means of introspective construction, is called intuitionistic mathematics. In many respects it deviates from classical mathematics. In the first place because classical mathematics uses logic to generate theorems, believes in the existence of unknown truths, and in particular applies the principle of the excluded third expressing that every mathematical assertion (i.e. every assignment of a mathematical property to a mathematical entity) either is a truth or cannot be a truth. In the second place because classical mathematics confines itself to predeterminate infinite sequences for which from the beginning the n th element is fixed for each n . Owing to this confinement classical mathematics, to define real numbers, has only predeterminate convergent infinite sequences of rational numbers at its disposal. Out of real numbers defined in this way, only subspecies of 'ever unfinished denumerable' species of real numbers can be composed by means of introspective construction. Such ever unfinished denumerable species all being of measure zero, classical mathematics, to create the continuum out of points, needs some logical process starting from one or more axioms. Consequently we may say that classical analysis, however appropriate for technique and science, has less mathematical truth than intuitionistic analysis performing the said composition of the continuum by considering the species of freely proceeding convergent infinite sequences of rational numbers without having recourse to language or logic." (Page 78)

Hence while intuitionistic mathematics appears to have much to recommend it from the point of view of secure foundations, the fact that it has to rely on a process that excludes so much of mathematics that undoubtedly works, ought to be at least a cause for concern. In order to meet the requirement of a paradox free mathematics, so much has to be excluded from 'mathematics', one is led to question the value of the exercise. To conclude that classical analysis has less truth than intuitionistic analysis makes one doubt this use of the term 'truth'.

A further objection can be made in the role of language for intuitionistic mathematics. According to Heyting the language used does not represent the mathematics. It is merely a method of communicating thoughts. Heyting then goes on to deny any transcendental nature to these thoughts. There must, therefore, be serious doubts about the possibility of sharing the same mathematical thoughts. If there is no certainty that the thoughts are common due to transcendental necessity, and no certainty that the language used will convey the thought, communication becomes at least unreliable.

According to the intuitionists, intuition is an introspective experience that provides an infallible guarantee of the truth of mathematical propositions. It has been suggested that intuitionistic mathematics can be a fruitful basis for mathematics education. Hence it is important to consider the way in which we use the term 'intuition' in mathematics education. Despite Heyting's denial, there is a definite platonic ring to the idea that introspection leads to universals, infallibly. We certainly do not use the term in the same way. Firstly, we recognise that there is a strong possibility that the student, when encouraged to use his/her intuition, will make a mistake. We are not depending on the infallibility of the step. Secondly we are, I suggest, expecting that our students will use their accumulated mathematical experience, often in an informal imaginative way, to determine how to proceed at a particular juncture. We are concerned with processes, hypotheses, suggestions, relational thinking, not an infallible awareness of truth.

2.4 Lakatosian View of Mathematics

Lakatos' view is not merely a fourth movement but is a radical alternative to the other schools of philosophy of mathematics. He characterises the work of the three traditional schools as attempts to reorganise mathematics on a Euclidean basis, and claims to show the inevitable failure of such a programme. His alternative is to reject even the attempt, and to adopt a quasi-empirical ideal for mathematics. His historiographical book "Proofs and Refutations" (Lakatos 1977) takes Euler's rule for the relationship between the vertices, edges and faces

of polyhedra, and traces the history of the development of a proof. In doing this he attempts to show, firstly, the heuristic as distinct from the deductive process of the growth of mathematical knowledge, and secondly the quasi-empirical nature of mathematical knowledge, in that counter-examples lead to the adaptation of the proofs, axioms or definitions in the theorem, i.e. the re-transmission of falsity rather than the transmission of truth.

In his article "A Renaissance of Empiricism in the Recent Philosophy of Mathematics" (1978) Lakatos explains the difference between Euclidean and quasi-empirical systems, and also the distinction between empirical and quasi-empirical. He writes:

"Whether a deductive system is Euclidean or quasi-empirical is decided by the pattern of truth value flow in the system. The system is Euclidean if the characteristic flow is the transmission of truth from the set of axioms 'downwards' to the rest of the system - logic here is an organon of proof; it is quasi-empirical if the characteristic flow is the retransmission of falsity from the false basic statements 'upwards' towards the 'hypothesis' - logic here is an organon of criticism. But this demarcation of patterns of truth flow is independent of the particular conventions that regulate the original truth value injection into the basic statements. For instance a theory which is quasi-empirical in my sense may be either empirical or non-empirical in the usual sense..." (Page 29)

Lakatos explains why Euclidean theories must fail, in another article, entitled "Infinite Regress and the Foundations of Mathematics" (1978). He writes:

"From the seventeenth to the twentieth century Euclideanism has been on a great retreat. The occasional rearguard skirmishes to break through beyond the hypotheses, towards the peaks of first principles, all failed. The fallible sophistication of the empiricist programme has won, the infallible triviality of Euclideanism has lost... These four hundred years of retreat seems to

have by-passed mathematics completely. Euclideans here retained their original stronghold. The mass of eighteenth-century analysis was of course a set-back. But since Cauchy's revolution of rigour they headed slowly but safely towards the peaks." (Page 10)

He continues:

"No Euclidean theory, however, can ever stand up to sceptical criticism. And the most incisive sceptical arguments against mathematical dogmatism came from the self-tormenting doubts of the dogmatists themselves: 'Have we really reached the primitive terms? Have we really reached the axioms? Are our truth-channels really safe?' These questions played a decisive role in Frege's and Russell's great enterprise to go back to still more fundamental first principles, beyond the Peano axioms of arithmetic. I shall particularly concentrate on Russell's approach, showing how he failed in his original programme, how he finally fell back on Inductivism, how he chose confusion rather than facing and accepting the fact that what is interesting in mathematics is conjectural." (Page 11)

Lakatos goes on to trace the history of Russell's work:

"We all know how the brief Euclidean 'honeymoon' gave place to 'intellectual sorrow', how the intended logico-trivialization of mathematics degenerated into a sophisticated system, including 'axioms' like that of reducibility, infinity, choice, and also ramified type theory - one of the most complicated conceptual labyrinths a human mind ever invented." (Page 18)

Russell himself wrote (1959):

"The splendid certainty, which I had always hoped to find in mathematics was lost in a bewildering maze." (Page 212)

Lakatos describes the options open to Russell. It is always possible to maintain a Euclidean programme, in the sense that one can hope that the axioms that one has had to use may later be shown to be reducible to self-evident axioms. Russell at first held this hope and then despaired of it. He rejected turning to postulationism, which involves shifting the self-evidence from the axioms to statements which one wants to derive. Russell turns instead to a dogmatic assertion of the inductive principle, as the only way out.

Lakatos' attack on inductivism in mathematics, following Popper's attack in science, is well known. Basic statements, no matter how many, can never prove an axiom.

Lakatos continues (1978):

"Let us now draw some of the conclusions Russell refused to draw. The infinite regress in proofs and definitions in mathematics cannot be stopped by a Euclidean logic. Logic may explain mathematics but cannot prove it. It leads to sophisticated speculation which is anything but trivially true. The domain of triviality is limited to the uninteresting decidable kernel of arithmetic and of logic - but even this trivial kernel might sometime be overthrown by some detrivializing sceptic criticism." (Page 19)

To return to Lakatos' quasi-empirical programme, we will look at the development of theories, the methodology of the programme, and the nature of falsifiers.

2.4.1 The Development of Theories

Lakatos (1978) compares the Euclidean and the quasi-empirical programme in the way that theories develop. He writes:

"The development of Euclidean theory consists of three stages: first the naive prescientific stage of trial and error which constitutes the prehistory of the subject; this is followed by the foundational

period which reorganizes the discipline, trims the obscure borders, establishes the deductive structure of the safe kernel; all that is then left is the solution of problems inside the system, mainly constructing proofs or disproofs of the interesting conjectures...

The development of a quasi-empirical theory is very different. It starts with problems followed by daring solutions, then by severe tests, refutations. The vehicle of progress is bold speculations, criticism, controversy between rival theories, problemshifts. Attention is always focussed on the obscure borders. The slogans are growth and permanent revolution, not foundations and accumulation of eternal truths.

The main pattern of Euclidean criticism is suspicion: Do the proofs really prove? Are the methods too strong and therefore fallible? The main pattern of quasi-empirical criticism is proliferation of theories and refutation." (Page 29)

2.4.2 Methodology

The distinction between the two types of theories is seen in their methodologies:

"The methodology of a science is heavily dependent on whether it aims at a Euclidean or quasi-empirical ideal. The basic rule in a science which adopts the former aim is to search for self-evident axioms - Euclidean methodology is puritanical, antispeculative. The basic rule of the latter is to search for bold, imaginative hypotheses with high explanatory and 'heuristic' power, indeed, it advocates a proliferation of alternative hypotheses to be weeded out by severe criticism - quasi-empirical methodology is uninhibitedly speculative." (ibid page 29)

2.4.3 Falsifiers

The role of formalization in mathematics, for Lakatos, is to extend the testability of a theory. He discusses falsifiers in the informal theories and the formal axiomatizations. He also discusses, in the extracts following, the question of a dividing line between science and mathematics. It is clear, and will be further discussed later, how significant these areas are for mathematics education, if one follows Lakatos' view of mathematics. He writes:

"If mathematics and science are both quasi-empirical, the crucial difference between them, if any, must be in the nature of their 'basic statements', or 'potential falsifiers'. The 'nature' of a quasi-empirical theory is decided by the nature of the truth value injections into its potential falsifiers. Now nobody will claim that mathematics is empirical in the sense that its potential falsifiers are singular spatio-temporal statements. But what then is the nature of mathematics? Or, what is the nature of the potential falsifiers of mathematical theories?... Let us first take comprehensive axiomatic set theories. Of course, they have potential logical falsifiers: statements of the form p & not- p . But are there other falsifiers? The potential falsifiers of science, roughly speaking, express the 'hard facts'. But is there anything analogous to 'hard facts' in mathematics? If we accept the view that a formal axiomatic theory implicitly defines its subject matter, then there would be no mathematical falsifiers except logical ones. But if we insist that a formal theory should be the formalization of some informal theory, then a formal theory may be said to be 'refuted' if one of its theorems is negated by the corresponding theorem of the informal theory. One could call such an informal theorem a heuristic falsifier of the formal theory."

(Page 35)

Lakatos summarises his philosophy of mathematics as follows:

"The gravest danger then in modern philosophy of mathematics is that those who recognise the fallibility and therefore the science-likeness of mathematics, turn for analogies to the wrong image of science. The twin delusions of 'progressive intuition' and of induction can be discovered anew in the works of contemporary philosophers of mathematics. These philosophers pay careful attention to the degrees of fallibility, to methods which are a priori to some degree, and even to degrees of rational belief. But scarcely anybody has studied the possibilities of refutations (in mathematics). In particular, nobody has studied the problem of how much of the Popperian conceptual framework of the logic of discovery in the empirical sciences is applicable to the logic of discovery in the quasi-empirical sciences in general and in mathematics in particular. How can one take fallibilities seriously without taking the possibility of refutations seriously?... It will take more than the paradoxes and Godel's results to prompt philosophers to take the empirical aspects of mathematics seriously, and to elaborate a philosophy of critical fallibilism, which takes inspiration not from the so-called foundations but from the growth of mathematical knowledge." (Page 42)

2.4.4 Lakatos' View and Mathematics Education

Before proceeding to examine some of the criticisms of the Lakatosian view of mathematics, some preliminary remarks are appropriate here on the significance for mathematics education. Lakatos is proposing a fallible view of mathematical knowledge, whereby what is interesting is conjectural. Inspiration is not to be found in looking at thoroughly axiomatized systems, but in the growth of mathematical knowledge. Speculative hypotheses, courageous conjectures and testable statements are the order of the day in mathematics. If we look at the way we teach mathematics, these ideas do not usually find a place.

Jack Easley wrote, in 1967:

"From the critics (of the new maths) we may also learn that the really new venture for mathematics curriculum reform lies in the area of heuristic procedures. If mathematics educators learn to apply the insights into mathematical inquiry which Polya and Lakatos have set forth, at the level in the teaching of mathematics on which the growing edge of the student's understanding happens to lie, the interest and achievement of students may be expected to increase markedly." (Page 228)

Even stronger support comes from Joseph Agassi (1980) whose article "On Mathematics Education: The Lakatosian Revolution" states in its title the position of the writer.

2.5 Some criticisms

It is interesting to note that Lakatos spent much of his time developing his Methodology of Scientific Research Programmes, whereas 'Proofs and Refutations' stands as his illustration of this theme in mathematics (Lakatos 1977). It is not developed to the same extent in his writing on mathematics. Indeed, Feyerabend in his book 'Against Method' (1975) hardly mentions mathematics at all, he refers to 'Proofs and Refutations' only once, and uncritically, and does not discuss Philosophy of Mathematics as such at all.

In his philosophy of science, whilst Lakatos rejects the idea that we can ever know that we are nearer to truth, he maintains some objective standards of rationality in two ways, by his distinction between internal history and external history, and in the identification of a research programme as a progressive one or a degenerating one. Both of these standards have been strongly criticised as being untenable (e.g. Bloor 1976). In his philosophy of mathematics, however, he appears to be quite happy to leave mathematics as conjectural. Indeed, he openly attacks the idea that truth has anything to do with mathematics. In the introduction to 'Proofs and Refutations' (1977) he writes:

"For more than two thousand years there has been an argument between dogmatists and sceptics. The dogmatists hold that - by the power of our human intellect and/or senses - we can attain truth and know that we have attained it. The sceptics on the other hand either hold that we cannot attain truth at all (unless with the help of mystical experience), or that we cannot know if we can attain it or that we have attained it. In this great debate, in which arguments are time and again brought up to date, mathematics has been the proud fortress of dogmatism. ... Most sceptics have resigned themselves to the impregnability of this stronghold of dogmatist epistemology. A challenge is now overdue." (Page 5)

The book is a case study of the history of mathematics as an informal, quasi-empirical affair, without his usual scientific distinctions between internal history, which is a reconstruction exhibiting the objective application of the particular methods of the particular research programme, and external history which is secondary, and looks at the reasons for errors, sociological factors not related to the rational growth of the theory etc.

Davis and Hersh (1981) have described the book as follows:

"It would be fair to say that in 'Proofs and Refutations' Lakatos argues that dogmatic philosophies of mathematics (logician or formalist) are unacceptable, and he shows that a Popperian philosophy of mathematics is possible. However, he does not actually carry out the program of reconstructing the philosophy of mathematics with a fallibilist epistemology." (Page 348)

The editors of Lakatos' book, John Worrall and Elie Zahar add their own comments in the form of Editors' footnotes scattered throughout the book. In these footnotes they reveal their unhappiness with Lakatos' world of uncertainty. For example, they write:

"This historical note, we believe, underplays the achievements of the mathematical 'rigorists'. The drive towards 'rigour' in

mathematics was, it eventually transpired, a drive towards two separate goals, only one of which is attainable. These two goals are, firstly, rigorously correct arguments or proofs (in which truth is infallibly transmitted from premises to conclusions) and, secondly, rigorously true axioms, or first principles (which are to provide the original injection of truth into the system - truth would then be transmitted to the whole of mathematics via rigorous proofs). The first of these goals turned out to be attainable (given, of course, certain assumptions), whilst the second proved unattainable." (Page 56)

This view is quite surprising given that they were editing Lakatos' work. The force of Lakatos' view of mathematics is that if there is anything that is of that nature in mathematics then it is at best uninteresting, and may at any time be refuted.

Michael Hallett (1979) attempts to take Lakatos' methodology of scientific research programmes into mathematics, in two articles in the British Journal for the Philosophy of Science. Hallett takes an idea of Hilbert's and adapts it, calling it Hilbert's criterion, for interpreting the expression 'mathematical progress'.

"Hilbert's criterion states that a consequence of T provides support for T if it is used to solve an 'important' problem and likewise was not used in the construction of T ." (Page 10)

He compares this with the methodology of scientific research programmes principle:

"The similarity with MSRP is quite clear. MSRP states that a consequence of T provides support for T providing it is true and was not used in the construction of T ." (Page 10)

Hallett claims that this criterion can be applied to the actual history of mathematics, and attempts to give an example using Cantor's introduction of transfinite numbers as an instance of mathematical

progress. Hallett needs something like Hilbert's criterion because an analogue of empirical data in science is needed in mathematics. Hallett uses problem solutions as this analogue. His programme is interesting but there are not yet enough developed examples on the lines of this article to discuss its validity. One wonders whether, had Lakatos lived longer, he would have returned to mathematics and attempted to complete his programme. It may be that he could not see a way to reflect his MSRP in mathematics and this is why he never returned to it. Hallett's article concludes with a view similar to that mentioned above (Grabiner 1974 Page 364), presumably a Kuhnian influenced view, that 'mathematics is that area of human activity which has at once the least destructive and still the most fundamental revolutions'. Hallett writes (1979):

"Another curious feature of theoretical rivalry in mathematics is that it appears to be resolved much more rapidly than rivalry in empirical science. Is this because there is no genuine rivalry between programmes in mathematics? Or is it that, because programmes do not need to wait for empirical support, heuristics are much more quickly exhausted in mathematics than in physics?"

(Page 155)

In a review article of the collected Philosophical Papers of Lakatos, Ian Hacking (1979) compares the significance of 'Proofs and Refutations' to some of the classics of philosophical literature. He also highlights the debt Lakatos owes to Wittgenstein. Hacking writes:

"It is seldom noted how useful it is to read the dialogue in company with Wittgenstein's Remarks on the Foundations of Mathematics (Lakatos put some rude and somewhat idiotic interjections about Wittgenstein into his later publications, but he read the Remarks carefully when writing 'Proofs and Refutations'). Where Wittgenstein gives hypothetical illustrations about following rules, diverging practices and concept formation, Lakatos gives real life examples. Wittgenstein's book is, in this respect, like a bestiary compared to Lakatos' natural history." (Page 391)

2.6 Conclusion

In this chapter the schools of thought in the Philosophy of Mathematics have been examined, including the use of extracts by many of the philosophers concerned. It can be seen that the Euclidean view cannot ultimately be refuted, in the sense that the apparent failures of those programmes grouped under the term Euclidean can be considered as temporary. For example, the logicians can maintain that eventually axioms such as the axiom of choice may be shown to be reducible to self-evident axioms. This belief in the certainty of mathematical knowledge forms the hard core of this paradigm. Lakatos' view is hence a real alternative, as its hard core is the uncertainty of mathematical knowledge and is thus a competing paradigm.

It has been suggested here that Lakatos appears to have been content to leave mathematical knowledge in a more uncertain state than scientific knowledge. As it stands, his view is, I suggest, in accordance with the 'relativist' view discussed in Chapter 1.

2.7 Summary of Section 1

In this first section, I have attempted to examine the current situation in the philosophy of mathematics, against its natural background of the nature of knowledge in general, and scientific knowledge in particular. I have attempted to show:

- (a) that views of the nature of mathematics are best seen as falling into two camps, the Euclidean camp and the Lakatosian camp;
- (b) that the Lakatosian alternative is in accordance with relativist views of knowledge ;
- (c) that maintaining a particular view of mathematics entails a view of knowledge that is appropriate;
- (d) that an epistemological commitment has great significance for mathematics education.

This last point forms the focus of the following section.

SECTION 2

THE CONSEQUENCES OF THEORY FOR THE PRACTICE OF MATHEMATICS EDUCATION

CHAPTER 3 - A REVIEW OF THE LITERATURE

In the last two or three years there has been a growing interest in what is being called the Theory of Mathematics Education. Before this, mathematics education texts would in general refer briefly, if at all, to the aims of mathematics education before passing on to aspects of the methodology. A brief discussion of this follows, in 3.1. The main part of this review is in three parts: The Philosophy of Mathematics as it Affects Teaching in 3.2, Alternatives of Ways of Teaching as Connected with Philosophy in 3.3, and Similar Work in Science Education in 3.4.

3.1 Aims of Mathematics Education

In an unpublished paper (Lerman 1980) I attempted to summarise the theoretical perspectives of some of the main mathematics education texts. In the absence of a thorough rationale for mathematics education - a task of which this thesis forms a part - and in recognition of the need to make some reference to why one teaches mathematics, many authors attempt to depend on criteria such as usefulness, beauty, or a tool of science (e.g. Watson 1976 page 122 and Mathematical Association 1974 page 186).

That such criteria are inadequate on their own is pointed out by, for example, David Wheeler (Land 1963):

"We cannot go to the stake for mathematics as 'the arithmetic of everyday life'; the amount of mathematics that is necessary, or even marginally useful, in everyday life is progressively diminishing - and it can certainly be taught to most children before the age of eight or nine." (Page 140)

In one sense this is true in that, for example, whereas one had to calculate the change to be given to a customer in a shop, today, simply pressing another button on the cash register causes the amount of change to appear on the display. In another sense quite the opposite is the case. Skills are even less adequate today than perhaps they were some

decades ago. The essence of the 'skills' required by the school-leaver is adaptability of their mathematical knowledge.

Others suggest that mathematics serves wider purposes, such as developing powers of reasoning, or as a means of identifying intellectual ability in general (e.g. Freudenthal 1973 page 62).

The former claim has had doubts cast on its validity by e.g. Morris Kline (1973):

"Another argument advanced by the advocates of the new mathematics is that their emphasis on logical structure teaches students to think deductively.... It is not the kind of thing that is useful in everyday life. The big problems and even the little ones that human beings are called upon to solve in life cannot be solved deductively." (Page 45)

A more telling criticism is made by Hirst (1974):

"... take something like the ability to solve problems. What is meant by this phrase? Obviously one begins by asking what problems: moral problems, scientific problems, mathematical problems? Clearly these are very different in nature. Can we assume that the ability to solve mathematical problems is the same as the ability to solve problems in morals? What is more, even to understand a scientific problem, as distinct from a moral problem, presupposes a great deal of scientific knowledge." (Page 20)

One often comes across instances of the latter claim, that mathematical ability is taken as a sign of general intellectual ability. The injustice is obvious.

Two important points arise from this kind of treatment of the aims of mathematics education. The first is that an inadequate justification leads to questions about the necessity of teaching mathematics at all. Far more important, though, and conceptually prior to the first, is that

one's view of what mathematics is, or why it should be taught, or not, and what purpose it serves, are major determining influences on the curriculum decisions made, methods of teaching adopted, research areas designated as important etc. Hence the importance of the kind of study carried out in this thesis, it is proposed.

Whether one accepts the necessity of arguments like Hirst's (1973 Ch. 3) about the way we structure knowledge, or not, there can surely be no doubt that quantification and other processes that are distinctly, although not necessarily exclusively, mathematical, are a sufficiently large a part of the nature of our existence and civilization that mathematics should be a fundamental part of education.

3.2 Philosophy of Mathematics as it Affects Teaching

As may be seen in historical studies such as Howson's (1982), attitudes to the nature of mathematics have always played a major part in the determining of the mathematics curriculum. The only difference today is that the situation in relation to what is mathematics is very complex, as discussed in Section 1 of this thesis.

In more recent times, considerations of the effect of philosophy of mathematics on teaching, although still few in number, have become explicit, as can be seen from the following studies.

As mentioned above, in 2.4.4, Easley suggests that applying the insights of Lakatos and others would increase students' interest and achievement (Easley 1967). Also Agassi (1980) refers to the effect of a teacher's philosophy of mathematics in his article.

Rene Thom (1973) has written:

"... all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics." (Page 204)

Reuben Hersh (1979) criticises formalism in school mathematics. He writes:

"The criticism of formalism in the high school has been primarily on pedagogic grounds: 'This is the wrong thing to teach, or the wrong way to teach'. But all such arguments are inconclusive if they leave unquestioned the dogma that real mathematics is precisely formal derivations from stated axioms. If this philosophical dogma goes unchallenged, the critic of formalism in the school appears to be advocating a compromise in quality: he is a sort of pedagogic opportunist, who wants to offer the student less than the 'real thing'. The issue then, is not, what is the best way to teach, but what is mathematics really all about... controversies about high school teaching cannot be resolved without confronting problems about the nature of mathematics." (Page 18)

In considering curriculum development in mathematics, Jere Confrey (1980) suggests that there are three theories of knowledge: absolutism; progressive absolutism, as exemplified by Popper; and conceptual change, based on the work of Stephen Toulmin (1972). She describes how each theory of knowledge brings with it a commitment to particular curriculum decisions. She writes:

"If the curriculum theorist accepts a curriculum change theory of knowledge in mathematics, then the question is what are the implications for the first task in curriculum design: the determination of content. This can be answered at two levels:

1. The basic tenet is that what one teaches ought to reflect the theory of knowledge which one thinks is most appropriate and adequate for that discipline. This means that one ought to select content which accurately portrays the particular discipline involved. For mathematics, it means that mathematics ought to be portrayed evolving, growing, and changing, not as static immutable truths.

2. An analysis of particular concepts in the discipline yields a variety of ways of conceiving of a particular concept, as well as a

variety of ways in which those concepts develop. Therefore, the curriculum theorist must also consider alternative conceptions of particular concepts to assess their appropriateness for inclusion as content." (Page 13)

This theme of conceptual change theory of knowledge and its influence on the mathematics curriculum is investigated further by Marilyn Nickson in her unpublished PhD dissertation (1981). Within the wider framework of the social context of the classroom, she discusses teachers' perceptions of the nature of mathematics. She examines the alternatives, and arrives at two, which she calls a positivist approach and a conceptual change approach. She then examines the curriculum process in the light of these two approaches, by researching two major school mathematics programmes, the School Mathematics Project and Nuffield Mathematics. She concludes that both are instances of a positivist approach, both are unsatisfactory, and much work is needed to develop alternative materials.

Together with Alan Bishop, she made an overall study of the social context of the mathematics classroom, presented to the Cockcroft Committee (Bishop and Nickson 1983). They point out:

"There is much public concern expressed about teachers' and prospective teachers', knowledge of mathematics, but research into the social context makes plain the need to focus more on the attitudes and perceptions of teachers with respect to the mathematical content of the curriculum." (Page 62)

The developments mentioned at the beginning of this chapter, that indicate a growing interest in what is being called the Theory of Mathematics Education, are in three main areas: research in the social context of mathematics education, as described above; research into a constructivist psychological view of learning, in particular in mathematics; and the initial discussions held at the Fifth International Congress on Mathematical Education in Adelaide, August 1984 (ICME V)

towards a new international group for the study of the Theory of Mathematics Education (TME).

At the 1983 meeting of the North American Chapter of the International Study Group for the Psychology of Mathematics Education, the theme of "Research in Mathematics Education from an Epistemological Perspective" was adopted for the conference. Several papers were presented relating directly to this theme, discussing the influence of a constructivist view of knowledge on the teaching of mathematics. This view will be examined in detail in Chapter 5, on learning theories. The relevance of this view from a philosophical perspective can be seen from the following extract from a paper presented by both of the editors of the proceedings of the above conference, Herscovics and Bergeron, at the 8th International conference for the Psychology of Mathematics Education, in Sydney, August 1984 (Herscovics and Bergeron 1984):

"A constructivist perspective of the teaching of mathematics focusses on the learner and the question "How can we guide him in the construction of his mathematical schemas on the basis of his existing knowledge?" is at the very heart of this approach. It is the prime concern which leads the teacher in the choice of his pedagogical interventions. Thus, to begin with, he has to determine what kind of knowledge can be used as a foundation for the building of the intended concept, and so ascertain that such a basis is present in the student. He must then take care that each step in his proposed construction is accessible to the pupil.

In contrast, a formal approach to the teaching of mathematics concentrates more on the transmission of knowledge than on its reconstruction by the pupil... A formal perspective of the teaching of mathematics is fostered by a formal perception of mathematics which is characterised by an emphasis on the form of mathematical expressions." (Page 190)

At a short conference held after ICME V, Prof. Hans-Georg Steiner initiated discussions on TME. The mini-conference was subtitled "A

needed comprehensive approach to basic problems in the orientation, foundation, methodology and organization of mathematics education as an interactive system comprising research, development and practice." This thesis is intended to be a contribution to this approach.

3.3 Alternative Ways of Teaching, as Connected with Philosophy

Little has been written which identifies what goes on in the classroom and relates it to a theoretical perspective, and in particular to a philosophy of mathematics. As described above, the reverse has been recognised, at least since Rene Thom, if not before.

Blaire (1981) discussed the relationship between philosophies of mathematics and what he termed perspectives of teaching. He identifies four movements of mathematics teaching:

- "(1) The teaching of mathematics as an art-form;
 - (2) The teaching of mathematics as a game (or family of games)
 - (3) The teaching of mathematics as a member of the natural sciences;
 - (4) The teaching of mathematics as technologically-oriented."
- (Page 148)

He goes on to consider and then reject the idea of mathematics as a language as a fifth perspective. He relates these movements to four schools of the philosophy of mathematics, and claims that teachers use one or other perspective as suits the subject matter or when influenced by other external pressures.

In an article published shortly after (Lerman 1983), I criticized this approach on the grounds that:

- (a) schools of thought in the philosophy of mathematics fall into two distinct movements, not four or five;
- (b) the influence of an epistemological position adopted on the teaching of mathematics is much stronger than Blaire makes out;

(c) an awareness of the consequences of epistemological commitments can lead to significant changes in school mathematics.

Brown and Cooney (1982), in an article concerned with research on teacher education, argue for a programme towards teacher education similar to that for learners of mathematics. They write:

"Mathematics educators are interested in how learners construct, interpret, and utilize the mathematical knowledge taught in classrooms. Similarly, we shall subsequently argue that such an orientation leads us to consider how teachers construct, interpret, and utilize the pedagogical knowledge taught in teacher education programs." (Page 14)

They mention further that the inclination of a teacher to teach in a particular way, to use one kind of content or approach or another, can depend significantly on their view, not just of the nature of mathematics, but of the nature of knowledge. However, the practical consequences of the possible different orientations is not developed in that article.

Plunkett (1981) discusses the idea that how one teaches mathematics is determined by one's view of the nature of mathematics. He identifies two views: the ontological one, whereby mathematics is taken to exist, and the alternative, which is to see mathematics as a creation of the human mind:

"with no existence other than in the minds of people..." (Page 46)

He goes on:

"If one sees mathematics as a body of knowledge (about the world, or platonic ideals, or formal systems) then one will be led to a view of teaching in which knowledge is passed (poured?) from the knower to the ignorant. What is to be passed can be prescribed in a syllabus and tested in an exam. The most convincing argument against the

value of this view, in my opinion, is that by and large people do not pass exams, and even those that do seem in a year or two to have been (cognitively) quite unaffected by their learning experience. If, however, one approaches mathematics as a set of human activities, one will try to introduce these activities to children. The aim will be to help them to experience them and enjoy them." (Page 46)

Within the scope of an article, the possibility for the development of ideas is limited, but nevertheless the nature of mathematical knowledge is more complex, as is the nature of the connection with the teaching of mathematics. Davis & Hersh (1981) point out that neither the 'ontological' nor the 'creation of the human mind' images of mathematical knowledge are adequate alone, but that what we have can best be described as a process of negotiation between the two views.

In a paper presented to the North American Chapter of the International Group for the Psychology of Mathematics Education, Cooney (1983) reports on a project being conducted in this area. The project concerns the beliefs about mathematics, and about the teaching and learning of mathematics, of four teachers, who are then observed in their teaching, after which interviews are held with the teachers and their students. His initial conclusions from the results so far, are related to the gap between the researchers' understanding of the concepts used, in the sense of the intended programme of the teacher educators, and that of the teachers. This, he suggests, is illustrative of a major concern for teacher education, namely:

"It appears that we cannot assume (do we?) that the conceptual models presented in our teacher education programs are the same models the students take with them into the classroom." (Page 169)

A related study, and part of the same project, was carried out by Alba Gonzalez Thompson, and reported recently (Thompson 1984). The investigation attempted to examine teachers' beliefs about mathematics and mathematics teaching and connections with teachers' behaviour, by a

case study of three junior high school teachers, observing and interviewing the teachers. She concludes:

"Although the complexity of the relationship between teachers' conceptions of mathematics and mathematics education cautions against making conclusive statements, the findings support the original assumption that led to this investigation. That is, teachers' beliefs, views and preferences about mathematics and its teaching, regardless of whether they are consciously or unconsciously held, play a significant, albeit subtle, role in shaping the teachers' characteristic patterns of instructional behaviour. In particular the observed consistency between the teachers' professed conceptions of mathematics and the manner in which they typically presented the content strongly suggests that the teachers' views, beliefs and preferences about mathematics do influence their instructional practice." (Page 125)

The work of Cooney referred to above, and this by Thompson, using a case study methodology, indicate the potential significance of a more thorough understanding of epistemological alternatives for mathematics and hence mathematics education, and of the nature of the connections with teaching behaviour. Within the context of such an analysis, case studies, group studies and others will, it is proposed, provide many valuable insights into the teaching of mathematics. What is missing, however, in the interesting and important work carried out so far, is an analysis of those alternatives, from the nature of mathematics itself, and resulting from this analysis a discussion of the nature of the connections with teaching. Without this, there is no complete picture of the way one can relate one person's views with another, or with the range of possibilities that exist for such views, and indeed the actions of the teachers.

3.4 Similar Work in Science Education

Within the scope of this thesis, and this literature review in particular, any investigation of parallel work in other disciplines must

necessarily be limited. It is, however, of interest to note one trend, at least, in curriculum discussions regarding science education. Two articles have appeared recently in the Journal of Curriculum Studies, both dealing with discovery learning in school science.

The first, by D. Royce Sadler (1982), looks at some of the limitations and implications of the principles of induction and hypothetico-deduction. He suggests that the move towards discovery learning in the 60's has largely stagnated, due to several factors, including the time-consuming nature of such a programme. He then examines some of the pitfalls of these two principles. He concludes:

"In this paper, I have tried to show that inductive and hypothetico-deductive reasoning, principles which once featured so prominently in a number of curriculum developments but are now more or less taken for granted in a much less extreme form, will not always proceed smoothly. Evidence has been drawn from the literature on the cognitive processes involved in human decision-making, judgement and evaluation to show that there do exist some identifiable human tendencies which can lead to bias in perception and interpretation of data. That these tendencies are not intuitively obvious can be explained, at least in part, by our innate capacity to cope with ambiguity in daily living, and by the checks and balances built into the procedures of normal scientific research.

At a time when there are renewed calls for a return to inquiry-based learning in the classroom, hard thinking tempered with some caution is required in finding out where and in what ways inductive and hypothetico-deductive principles can be used to advantage." (Page 50)

Whilst this analysis of the situation in science education is extremely important, what is missing is the theoretical perspective which has led to the introduction of discovery learning into the classroom, whether of the inductive or of the hypothetico-deductive form, which view of the

philosophy of science is reflected in this approach, and hence where next to proceed, i.e. the consequences of the views.

This kind of analysis is provided by the second article, by Harris and Taylor (1983). They examine the inductive tradition from Bacon to Mill, and the critique of Hume. They propose that the hypothetico-deductive tradition was given its main impetus by Popper, although they point out the difficulty of following a strict Popperian programme in schools. They next examine two different curriculum programmes of secondary science, for the ways that they use discovery learning, and they also identify the links with a child-centred view of education as a whole. They conclude:

"Paradoxically, it is the education of science teachers rather than the education of children where the philosophy of science matters. We would like to encourage teachers to be more critical of the assumptions on which many science curricula are based. We suggest that science teachers be encouraged to take an interest in the philosophy of science. It is our hope that they may approach their task with a greater appreciation of the intentions of curriculum-designers and be less likely to accept naive approaches to discovery learning - less likely to waste the time of children with undirected experimenting, and less likely to deceive them with oversimplified justifications." (Page 288)

The former analysis is analogous to the criticisms of school mathematics without the fundamental challenge of the implicit formalist or platonic assumptions underlying the syllabus and the ways of teaching. The latter analysis does not make the same mistake, and hence is a more thorough perspective on the issue of discovery learning in school science. Just as in mathematics, this kind of approach, viewing the issue of science as a whole, and not piecemeal, is essential when considering aspects as fundamental as the role of discovery learning, and one can expect an increase in the literature on this.

3.5 Conclusion

Clearly there is a growing interest in relating the way of teaching mathematics to the beliefs about mathematics held by the teacher, but little research, either of the theoretical alternatives as regards views of mathematics, or of the strong connections between views, curriculum choices, and ways of teaching, or indeed attempts to identify teachers' views and classroom styles. This thesis is an attempt to further research in this area, by attempting to establish the philosophical alternatives, identifying the nature of the connections between epistemological view and practice, and by initiating attempts at identifying both in practising teachers.

CHAPTER 4 - THE RELATIONSHIP BETWEEN TEACHERS' VIEWS AND TEACHERS' ACTIONS

The examination and discussion of epistemological alternatives in mathematics, in Section 1, has led to the proposition that there are two identifiable positions, the Euclidean programme and the relativist, Lakatosian view. It was also suggested that the adoption of an epistemological position may have a direct bearing on important and fundamental aspects of mathematics education, in particular syllabus content and teaching behaviour. This latter will be examined here,

In 4.1 the nature of the connection between views and actions is discussed. In 4.2 it is proposed that in this context teachers' actions can be described by a one-dimensional continuum, whose ends reflect the philosophical alternatives as outlined in Section 1. These two ends are outlined in 4.2.1 and 4.2.2. In 4.3 four situations that occur in mathematics teaching are described, with the intention of highlighting, through them, the proposed differences in possible teacher actions.

4.1 The Connection Between Views and Actions

Teachers and teaching may be seen as the mediating stage between the context, which is specifically the pupils, but which clearly reflects the school, parents and society as a whole, and the intended result which is, specifically, here, the mathematics learning of those pupils.

Thus the influences on teachers are from varied sources and are often conflicting ones. They include pupils demanding interesting relevant work, whilst expecting difficult and tedious work; parents expecting success in external examinations, relevance to the real world of work, in order to help their children obtain the few jobs that there are available, and also perhaps hoping for some similarity with the mathematics they knew at school; local education authorities expecting cost-effective examination entrance and success rates, backed by the headteacher's concern regarding examination results being a significant factor in the standing of the school in the area; other departments

within the school requiring particular mathematical skills at stages that suit their own courses, but that often conflict with the mathematics department's own programme, and also the common problem of pupils being able to recognise that a learned procedure can be used in other subjects, and the implied criticism of the mathematics department as a consequence; commerce and industry criticising school for the irrelevance of the school mathematics syllabus content and the poor ability of school mathematics graduates, even those that have secured an external qualification in mathematics; and many others. Teachers have to respond to these demands, in attempting to secure the mathematical learning of their pupils. It is proposed here that the beliefs and values that teachers hold, regarding education and regarding mathematics, are significant factors in determining how teachers respond to these pressures in the context in which they work. In relation to mathematics, it has been proposed that an epistemological view involves a commitment, and that teachers respond and react, at least partly, as a consequence.

It must be said that this is not necessarily the case, in the sense that the relationship between views and actions, taken together with the other factors relating to teacher behaviour, may be different. It may be the case that views do not determine actions, rather that they are a gloss to the actions of a teacher. A teacher may behave in particular ways that are determined perhaps by pragmatic decisions, and the views either bear no relation to the actions, or are construed as to represent or justify the actions. This point is clearly vital to this thesis, since a case is being made that the two alternatives for teachers' views are reflected in the alternatives for teachers' actions. The arguments put forward here are in support of the idea that there is a connection, and that the connection is a direct one, from views to actions and the reverse, in relation to the nature of mathematics and the teaching of mathematics. Were this to be a discussion about theories of education in general, and the practice of teachers, the problem would perhaps be much greater. Similarly if one were discussing a person's view of the meaning of life, and how this affects, determines, or is determined by, the way that he or she lives their life, the connection would be complex

and thus difficult to evaluate. However, in this present work, one is dealing with a subject, mathematics, that has a considerable body of well-established results, generally accepted by the community, and in that sense objective. There are certainly values involved in one's views of the nature of mathematics, but it is not of the same kind as the two instances above, which are in a sense, nothing but values, views of 'good', 'just' etc. Mathematics teachers, and of course others, have to deal with this collection of stuff we call mathematics, whatever it is, and indeed are not generally called upon to consider their views or values in relation to mathematics. Thus the way one has learnt mathematics, the way one's teachers presented it, the degree to which one enjoys it, the way one relates to the certainty or otherwise of it, the way one uses it or perceives how others use it, the extent of one's reading around it, the way one uses it in school in relating to pupils and indeed staff etc., are the determinants of a person's view of the nature of mathematics, whether coherent or conscious or not. Concepts, ideas and theories are determined by their use, by the complex rules, customs and procedures of the 'language game' of mathematics. Thus, theories and practices are necessarily inter-related.

As discussed above, there are many other factors that influence teachers and contribute to determining teacher behaviour, for example the authority relationship within the mathematics department and the school. The complexity of the influences that affect mathematics teacher behaviour thus forms a central focus of the study in Section 3.

4.2 A Continuum of Mathematics Teachers' Actions

If schools of thought on the nature of mathematics can be seen to fall into two camps, the teaching behaviour of mathematics teachers cannot be so simply divided. Nevertheless, it is hypothesised that the philosophical positions described above can be seen to be reflected in two stylized teaching patterns, which, whilst not representing the behaviour of any single teacher, can serve as opposite ends of a continuum, within which teachers will fall. It is proposed that a meaningful perspective on mathematics education is provided by this

model. The two hypothesised ways of teaching that serve as the two ends of the continuum will now be described in some detail, followed by a discussion of four hypothetical situations that can and do occur in mathematics education, which will help to highlight the significance of this perspective of alternatives of ways of teaching mathematics.

4.2.1 Mathematics from a Euclidean View

It may be seen that the adoption of a Euclidean approach to mathematics implies the tendency to see the teaching of mathematics as a process of bringing pupils to see the certainty and timelessness of its results, and the deductive nature of its methods. This means that, in doing mathematics, the correct methods of deduction must be applied, and that provided the questions are set out correctly, the desired result of the right answer will be achieved. There is no 'purpose' in the sense of a particular problem to be solved. It is the method that is of central significance, or the particular section of the body of mathematical knowledge that is considered as essential knowledge for the school mathematics syllabus. Provided the methods and the content have been thoroughly learnt, and tested by repeated exercises, mathematics has been successfully conveyed. The psychological overview is one of the conveyance of knowledge, in its most efficient form. Mathematics is seen as a steadily accumulated body of knowledge, linear, hierarchical, dependable, reliable and value-free. Concepts do not develop, they are discovered.

Syllabus content, by this view, is a question of selecting a broad range of subjects, so as to give a foundation in many areas of mathematical knowledge, emphasising the common structure and methods used throughout mathematics. The approach of the teacher is essentially geared towards an esoteric view of mathematics. Pupils are given to understand that not only is the teacher in the position of having the knowledge that the pupil is required to learn, but that the teacher also has the knowledge of the relevance, significance, justification and reason for learning those things. Hence it is not appropriate for the pupil to ask why a particular topic is on the syllabus, what relevance a piece of

mathematics has to the real world, or what applications there are to parts of mathematical knowledge. This is perhaps why so many teachers find it so uncomfortable to be asked such questions. It is somehow not satisfactory to answer in the mode described here, but there is no other answer. The teacher is conveying the view that one learns the systems of mathematics and then one learns to apply them, some time in the future, during employment, retraining in industry or some other later process. This probably characterises most university courses as well. It is certainly not the role of school mathematics to concern itself with uses or applications. The topic of percentages, for example, is taught in the classroom by teacher exposition, practised by pupils through many exercises, and it then can be applied by pupils in any one of the various situations in which it is applied: bank interest; VAT calculations; errors in engineering calculations etc. This is the view of the teaching process as a consequence of seeing mathematics as a body of knowledge, in this Euclidean sense.

Since mathematics is independent of the context of its discovery, or the sources of inspiration of the discoverer, those being the external history of mathematics, the more familiar and acceptable sociology of mathematics, presenting the problem to pupils in an historical context is not relevant. If it is included, it is merely in the form of spice in a cake.

On the other hand, there are positive consequences to the adoption of this perspective of mathematical knowledge in the sense of the achievement of specific aims, or at least designing research towards the improvement of the realisation of specific aims. An examination that is geared towards new forms of questions whose essential style and content are familiar from every previous examination, is far easier to prepare one's pupils for sitting, than one that may consist, partly at least, of open-ended investigations. Education towards the successful application of heuristics, problem-solving methods, is well known to be far more problematic than a research programme aimed at the successful teaching of fraction addition, for example (e.g. Lowenthal 1984).

4.2.2 Mathematics From a Lakatosian Alternative View

It is interesting to note that, although Lakatos does not appear to have written anything directly related to school mathematics education, his book 'Proofs and Refutations' (1977) is written as a classroom dialogue, in which the ideas of the students and the discussion between them describes the stages of the development of a proof for Euler's relation for the faces, edges and vertices of polyhedra. The teacher's role is to propose the initial problem, and adjudicate in the disputes, at times summarise the arguments put forward, but is not in the position of knowing the answers any more than the students. This leads one to wonder what are the consequences for the teaching of mathematics if one accepts his description of the nature of the process of the growth of mathematical knowledge, and the relative state of the notion of certainty or proof in mathematics.

The adoption of this view implies the tendency to see the teaching of mathematics through a problem-solving perspective. Without the context or purpose of particular parts of mathematics, that is, the problems that inspired that work, it is at least very difficult for pupils to find meaning in mathematics. The psychological overview is one of pupils constructing their own knowledge, by comparing a new problem, idea, object, hypothesis against their existing experience, and conceptual system. Hence pupils, by this hypothesised view, must be encouraged to propose ideas and suggest methods; they must be led to test these conjectures themselves, to try to generalise their methods, compare them with other possibilities and search out other problems of a similar nature that may have been previously solved. In this view, unless the motive or goal in the form of a particular problem, has been set, mathematical methods or knowledge of mathematical systems is isolated, lifeless and, most important in this context, not representative either of the nature of mathematical knowledge, or its growth, or indeed of the learning process.

Construction of the syllabus, according to this view, is also quite a different process. Mathematical knowledge as a whole can be seen as a library of accumulated, tested and relatively reliable results of the work of people inspired by interesting problems, abstract, real world, or somewhere between the two. This library has to be theoretically accessible to everyone, for one to be able to say that mathematics education is going on, to whatever degree of ability and understanding one's pupils are capable of attaining, i.e. as appropriate in particular instances. Hence the emphasis is on processes of mathematical thought, and content that is appropriate for the development of these. When a pupil is aware, by his or her own attempts to solve a given problem, that information is needed, or other solutions of a similar problem would help, then accumulated knowledge becomes relevant and accessible.

The history of mathematics, by this hypothesised view, becomes an integral part of learning mathematics in school. The historical development of the topic, the original problems that inspired the work, provide collective meaning, in the sense of the understanding and progress of the mathematical community and society in general (e.g. Kline 1972), and can often provide or assist individual meaning, as in the case of the geometrical interpretations of the foundations of the calculus, for example.

School mathematics is seen as a model of all mathematical activity, and not as a matter of acquiring certain skills and knowledge, which are then suddenly applicable.

4.3 Four Situations in Mathematics Teaching

Within the context described here, it is instructive to examine some issues in the teaching of mathematics and identify the different strategies or attitudes that could be adopted. The alternatives will be outlined, and will be seen, it is proposed, to clarify the whole view of teaching behaviour of this thesis. The first is a consideration of what one does with novel pupil ideas, discussed through a fractions problem. The second is an illustration of the significance of one's views when

making curriculum content choices, through the question of the role or place of geometry in the school curriculum. The third deals with an aspect of teacher-pupil interaction, namely 'discussion', and identifies different approaches to what it might mean, from different theoretical perspectives. The fourth situation looks at an element of school mathematics considered to be most important, investigations, and the possible pitfalls if one is unaware of the significance of investigations. These four situations are not intended to be exhaustive of all mathematics teaching, but are four of the broad types of issues that are faced in mathematics education.

4.3.1 A Fraction Problem

In an unpublished paper, Alan Bishop (1977) reports on a lesson he gave to pupils familiar with fractions, but not experienced in combining them. He asked the class for a fraction between $1/2$ and $3/4$. A pupil answered " $2/3$ ", and when asked to give his reasoning, the pupil replied that 2 (the numerator) is between 1 and 3, and 3 (the denominator) is between 2 and 4. At this stage it may be seen that the teacher has two choices. The first includes such options as saying that the method is wrong, although that particular answer is right; pointing out that another example such as $1/2$ and $1/3$ will not yield to that method; asking the class if anyone has a better way of solving the problem; asking the class for a counter-example; noting that the particular method works for that particular sort of example, but that there is a much better method that works for all examples, etc.

All these are one choice, namely that there is a particular method, or there are particular methods, for solving that problem, that the teacher knows the correct method or methods, and that the pupil's role is to either arrive at that method, or to wait long enough so that the teacher will show the class how to solve the problem.

The second choice open to the teacher is, firstly, to perhaps express inner surprise that the answer is correct although the method was unexpected; second, to make the class check that the answer is correct,

and to congratulate the pupil; thirdly, to encourage the class to look for other examples and to use the method to solve them, until counter-examples appear, or in this case instances that do not appear to yield to the suggested method, to attempt to extend the method, adapt it, alter it, or if necessary reject it. It happens that in the example given here, the method can be extended, with a collection of rules to cover the different sorts of possible questions that may arise.

In this choice, the teacher is aiding the class to do mathematics, to make and test hypotheses, generalise and so on. By receiving credit for the novel suggestion the class gain the impression that mathematics is something they can do, and not that they have to wait to be shown the given, fixed, correct methods by the teacher. The teacher's question is a genuine one, in the sense that answers are treated as valid by the teacher, it is not a matter of guessing what answer is in the teacher's mind. It can be seen how this exchange between teacher and pupils reflects the process of the growth of mathematical knowledge according to the view described as Lakatosian (Lakatos 1978):

"The basic rule... is to search for bold, imaginative hypotheses with high explanatory and 'heuristic' power, indeed, it advocates a proliferation of alternative hypotheses to be weeded out by severe criticism..." (Page 29)

4.3.2 The Debate over Geometry in the School Curriculum

The place of geometry in the school mathematics curriculum remains a subject of much discussion (e.g. Zeitler 1983, Fielker 1983, Tahta 1980, Gattegno 1980, Hilton 1984). The term 'Euclidean' has been used in this thesis to describe those philosophies which aim at the Euclidisation of all of mathematics, i.e. which aim to eliminate all contradictions and other problems in the foundations of mathematics by basing mathematical knowledge on axioms which are intuitively obvious, or at least consistent, and by using methods of deductions that avoid such difficulties. Euclidean geometry provides the model because of its apparent structure of truth flowing downwards from intuitively obvious

axioms, and definitions. Clearly then, geometry itself is a revealing area of discussion for school mathematics, in relation to views of the nature of mathematics. One is forced to consider the nature of proof, its role in mathematics in general, and its place in school mathematics. One confronts in a very direct way the problem of stages of development of children's intellectual growth, and when for example, they might be able to be expected to understand the significance of proof. Different examination syllabuses place emphasis on different approaches to geometry, namely the traditional Euclidean approach or through transformation geometry. The choice made by the teacher as to what is taught and how, will depend significantly, it is proposed here, upon the view held of mathematics, and the role of geometry in the teaching of mathematics.

The school which is the subject of the first-stage study described in Section 3 is a case in point. The Head of Department did not like matrices and transformations and hence chose an 'O' level syllabus that treated geometry in a traditional Euclidean way.

In the literature, opinions vary between, e.g. the views of Jean Dieudonne (in Howson 1981):

"And if the whole programme I have in mind had to be summarised in one slogan it would be: Euclid must go!" (Page 102)

"With regard to geometry, I understand that much research and experimentation has been going on... concerning the methods by which this teaching of geometry as a part of physics, so to speak, can be conducted. I think this development should be highly encouraged, provided it puts the emphasis not on such artificial playthings as triangles, but on basic notions as symmetries, translations, compositions of transformations, etc." (Page 104)

to those of David Fielker (1983):

"Early ideas about number are based on spatial experiences. Our models for number, for operations, for place value, for algorithms and for extensions to fractions, decimals and integers are invariably visual and therefore spatial. Even some of our algebraic ideas are based on diagrams. And most work in 'modern' algebra relies on diagrams or on relations and transformations in space. Even notation is geometrical!

So geometry is perhaps more basic than arithmetic or algebra. It is also much easier. At least, concepts about space are easy."

(Page 26)

On the basis of these extracts, it would appear that David Fielker's interest in geometry in the school mathematics curriculum is because of the power and extent of the intuitive ideas in geometry. Dieudonne's concern appears to be a strong formalism, in which intuitively powerful but logically problematic objects such as triangles are to be rejected in favour of notions such as symmetry which are structure-based.

It may be seen that there is a strong similarity rather than difference, between the traditional Euclidean school geometry and the Ehrlangen programme-based ideas proposed by Dieudonne, i.e. that geometry characterises mathematical knowledge, and the argument is down to a platonic programme or a formalist one. David Fielker's view, however, is quite different. He sees the immediacy of the concepts of geometry for children, and the wide use of spatial notions provides many meaningful points of contact for pupils, in enabling them to pick up and play with mathematical ideas and methods. The terms chosen are intended to reinforce this view of the power of geometry for school mathematics because this is quite literally what one can do with 'artificial playthings' such as triangles. These two views are quite different in the way that they treat the role of parts of mathematical knowledge in relation to school mathematics, and can be seen, it is proposed, to reflect the alternative view of the nature of mathematics.

4.3.3 Discussion in the Mathematics Classroom

Some considerable interest has been shown in the meaning and significance of one part of paragraph 243 in particular, in the Cockcroft Report (1982):

"Mathematics teaching at all levels should include opportunities for... discussion between teacher and pupils and between pupils themselves." (Page 71)

This interest includes a publication by the Association of Teachers of Mathematics (ATM 1984), and a number of sessions of the British Society for the Psychology of Learning Mathematics, in 1984.

'Discussion' is defined in the Pocket Oxford Dictionary, Fifth Edition, as "Exchange or compare opinions upon (subject)". Whilst no teacher would be likely to disagree on the need for verbal interchange in the mathematics classroom, for a discussion to take place implies that more than one person in the classroom has a valid opinion on the subject, and that an exchange or comparison of opinions can take place. Again, most, or at least some, teachers would agree that discussion can and should take place between pupils, but to recognise that discussion can also take place between teachers and pupils is another matter.

If the prevailing view is that the teacher has the knowledge, and it is the role of the pupils to glean that knowledge, 'discussion' as defined above does not describe what takes place in the mathematics classroom. Pupils do not have views or opinions about mathematical knowledge, nor in fact does the teacher. There is mathematical knowledge, the teacher possesses that knowledge, and his/her job is to convey it to the pupils.

If, however one recognises the nature of pupils' novel ideas, as described in 4.3.1 concerning finding a fraction between two others, and in many other instances in the literature (e.g. Duckworth 1972), there is real discussion taking place in the classroom, examination of pupils'

ideas on an equal footing, pupil with teacher, and negotiation of meaning between the participants in that lesson.

It is not clear if the members of the Cockcroft Committee were aware of the implications of that clause in paragraph 243 that refers to the need for discussion to take place in the classroom, but it is suggested here that it serves as another illustration of the significance of an adopted epistemology for the strategies and criteria used in mathematics education.

4.3.4 Attitudes to Investigations

Problem-solving has been described as the focus for mathematics education in the 1980's by the National Council for Teachers of Mathematics in the USA. The journals of mathematics education are indeed full of articles describing classroom experiences of investigations, or ideas from departments of education. Hersh has indicated (1979 page 33), that unless one tackles the philosophical assumptions underlying teachers' attitudes, changes in the curriculum do no more than shore up the breaches. The result of attempting to introduce investigations into the curriculum of teacher education, or into mathematics departments in schools, without a full perspective of the implications, is often met with resistance, or attempts in the classroom that do not succeed. Two instances of this are described here.

I observed a lesson given by an experienced teacher, with an able class of 15 year-olds, in which the teacher, under instructions from the Head of Department, was to do an investigation. He chose an investigation described in a journal (Corps 1983), and prepared the mathematical background and implications of the problem in great detail before the lesson, and presented the class with the problem at the start. He then proceeded to wander around the class giving guidance, in the form of "Carry on that way and you'll get the right answer" or "There is an answer, I assure you, it's here on this piece of paper" or "No not that way, try this", until after about 15 minutes one pupil asked, as if for

the whole class, for the teacher to tell the method and answer. The teacher resisted, and managed to encourage the class to continue on for themselves, but at the end of the lesson, the teacher expressed to me his doubts about the value of the exercise, whether anything different or lasting had taken place in that lesson, and whether perhaps it was better to just get on with the usual mathematics. It was clear that he had not been given any guidance on why a class should do investigations at all, or what the pupils and he would gain out of it. In a different situation this could have been a useful opportunity for a mathematics department to discuss the implications of the exercise, and learn from it.

A similar pupil reaction is recorded in the editorial of the first edition of the magazine "Investigations" (Smile 1984):

"More and more teachers are wanting to start using investigations. But there are problems. Children say... What's the point? ... What should I do?... What's the answer?... This is silly!... I want to do proper maths!... This won't help me get a job!" (Page 1)

From the Euclidean view of mathematics, investigations are likely to be seen as playing at doing mathematics, or as suitable activities for just before Christmas, or after school examinations in the summer. At best, solving problems may be seen as good training for answering difficult examination questions. Open-ended investigations, however, are often considered too open for pupils to make any recognisable progress, or to see any results. There is of course the important and difficult area of how to assess investigations. However, investigations can be seen to represent an alternative perspective on the learning process and on the nature of mathematical knowledge. Thus one sees the relevance and importance of investigative work, and the need to develop methods of working for the teacher and the pupils.

4.4 Conclusions

In this chapter I have attempted to show the connections between views of mathematics and ways of teaching, this latter being characterised by a continuum of teaching behaviour, the ends of which reflect the two perspectives of the nature of mathematics. In the next chapter some recent and relevant developments in the psychology of mathematics education will be examined.

CHAPTER 5 - SOME RECENT DEVELOPMENTS IN MATHEMATICS LEARNING THEORIES

Research in the psychology of learning mathematics is clearly not some independent occupation, unrelated to the views and prejudices of the researcher. An image of human behaviour as mechanistic, or alternatively as organismic, will determine the hypotheses, methodology and conclusions of any research on learning. Development of a theory of mathematics education must be influenced by, and in turn influence, research on the psychology of mathematics learning.

In this chapter, a recent development in research into the psychology of learning mathematics will be discussed, namely the constructivist view. In 5.1 a brief overview of research in the recent decades is given, with a view to identifying the stimulus for new directions of interest. Section 5.2 describes the development mentioned above. This view will be seen to have close links with the philosophical perspective developed in this thesis.

5.1 An Overview of Research Directions

Most research in psychological aspects of mathematics education in recent times has been from a cognitive perspective rather than a behaviourist one. The emphasis has shifted from the 1950's and before, when much research focused on applications of behaviour analysis and reinforcement theory in the classroom. Recent decades have rather focused on the non-observables of thinking, reasoning etc., exhibiting a concern with what is to be learnt, with hierarchies of difficulty and complexity, and with the development of teaching materials and methods of assessment to best aid the progression of the individual up the hierarchies of mathematical concepts. The predominance of the cognitive perspective is described, for example, by Greer (1981):

"Since then cognitive theories have virtually wiped the board."
(Page 19)

The achievements of cognitive psychology have been characterised and also criticised by Resnick and Ford (1984):

"The old learning psychology had much to say about how to arrange conditions of learning; but it was weak in its ability to describe the content of learning. Cognitive psychology, by contrast, offers rich descriptions of the content and processes of performance in a subject matter. But up to now it has said almost nothing about how competence is acquired." (Page 244)

Greer too, indicates the limitations on much of research (1951):

"Having achieved predominance, Cognitive Psychology is now undergoing something of a crisis of confidence." (Page 19)

"While there is plenty of current research on mathematical topics... there is little by way of general theory. Signs of increasing interest among cognitive psychologists in mathematical thinking can be detected (though it still looks suspiciously as if they are motivated more by convenience than by interest in the subject itself)." (Page 20)

Today, several years on from Greer's observations, there is much genuine interest in mathematical thinking.

Robert Davis (1983) describes this as a paradigm shift:

"An earlier paradigm focussed on teaching and learning whereas the emerging paradigm focuses on the processes of thinking about mathematical problems." (Page 254)

Elsewhere, Davis (1984) outlines some of the weaknesses of the old paradigm, and signs of optimism in the new. In doing so, he identifies the essential starting point for the new directions in psychological studies of teaching and learning mathematics:

"...by omitting the postulation of a theory (or conceptualization) of human information processing, the 1950's paradigm attempted what has never really succeeded: an empirical science without a postulated conceptual foundation." (Page 372)

In a review of a recent book by Steffe, von Glasersfeld et al. (1983), Carpenter describes the difference between the two major research perspectives (1985):

"Probably the most important approach to the study of children's thinking is information processing... The basic assumption underlying information-processing theory is that thinking involves symbolic manipulation. The goal is to construct precise models of sequences in which discrete pieces of information are processed for a particular problem situation. Although the information-processing approach is used to model complex thinking processes, it is essentially mechanistic in nature. Steffe and his associates view thinking processes from an organismic perspective and describe thinking in global terms. It is not simply that their analysis is not as precise as an information-processing analysis; they do not believe that thinking can be broken down into component parts that can be specified as a sequence of discrete acts. They rely on constructs like intent and meaning, which are not readily analyzed into information-processing terms." (Page 74)

Carpenter describes the foundation of the differences between the two approaches as being a philosophical one and suggests that usefulness in explaining children's thinking may be the criterion for which model to adopt. 'Usefulness' is, however, a notion that involves making judgements, an exercise which is involved with values, beliefs and philosophy. This does not invalidate the idea of comparing the two approaches using a number of criteria, including usefulness, but merely identifies the limits of the comparison.

The latter approach, that of the organismic perspective, or constructivist view, will now be examined in more detail.

5.2 Constructivism

The central theme of the constructivist view has been described by von Glasersfeld (1974) as:

"A person's representation of the environment, his/her knowledge of the world, is under all circumstances the result of his/her own cognitive activity." (Page 22)

Before examining some of the research from this perspective, it is important, given the nature of this thesis, to trace the origins of constructivism in Piaget's own writing.

For Piaget, constructivism describes the process whereby new structures are formed, and hence forms a central theme of his genetic epistemology. He summarises his theory, and attempts to demonstrate the failures of alternative epistemologies, in the final chapter of his book "The Principles of Genetic Epistemology" (1972). He states the problem as:

"... does genesis correspond to a hierarchy or even a natural interdependency of structures; or does it merely describe the temporal process by which the subject discovers these structures as pre-existing realities? The latter alternative involves the view that these structures are preformed: either in the objects of physical reality, or as a priori in the subject himself, or in the ideal world of possibility in a Platonic sense. Now, through its analysis of genesis itself, genetic epistemology has tried to show the inadequacy of these three hypotheses, and to make a case for the view that genetic construction in its wider sense is an effectively constitutive construction." (Pages 88-89)

Starting with the Platonic view, Piaget takes the case of mathematical knowledge, and the idea that it exists for all time independently of mathematicians. He writes:

"Both history and psychogenesis seem to show, first, that the hypothesis of such a permanent existence... adds nothing to logico-mathematical knowledge itself and in no way modifies it; and, second, that the subject does not possess any cognitive procedure enabling him to arrive at such entities, assuming that they exist; the only known methods of logico-mathematical knowledge being those which occur in its construction and are thus self-sufficient."
(Page 89)

As regards the view that structures are preformed in the objects themselves, Piaget writes:

"Objects certainly exist, and they involve structures which also exist independently of us. But objects and their regularities are known to us only in virtue of operational structures which are applied to them and form the framework of the process of assimilation which enables us to attain them." (Page 91)

In dealing with the third possibility:

"The a priorist hypothesis, which locates such predetermination within the subject rather than in objects... It seems genetically clear that all construction elaborated by the subject presupposes antecedent conditions." (Page 91)

Piaget gives one or two examples of a priori forms which proved too strong and were shown to be constructive progressions of more basic structures. He thus claims:

"So it seems that if we wish to arrive at an authentic a priori, we must progressively reduce the 'intension' of the initial structures until what remains qua antecedent necessity is reduced to a simple functioning. It is from the latter that these structures originate... Clearly, then, this functional a priorism in no way excludes but rather lends support to the theory of continuous construction of new structures." (Page 91)

These arguments of Piaget's, putting forward his justification of a constructivist epistemology are summaries of developed arguments from elsewhere in his writing. The implications of Piaget's work for education has been extensively researched, developed and criticized. Recent research, however, has taken up the consequences of Piaget's epistemology rather than the details of his description of developmental stages.

Von Glasersfeld (1983) writes:

"This view of knowledge, clearly, has serious consequences for our conceptualization of teaching and learning. Above all, it will shift the emphasis from the student's "correct" replication of what the teacher does, to the student's successful organization of his or her own experience." (Page 51)

He criticizes a large part of educational research as:

"... setting tasks, recording solutions, and analysing these solutions as though they resulted from the child's fumbling efforts to carry out operations that constitute an adult's competence." (Page 61)

He proposes instead that the "teaching experiment" in particular, as developed by Steffe, appears most fruitful for fostering self-awareness which he suggests is the key to successful operational constructions. He examines the method of clinical interviews, and claims that what is happening is that the researcher is developing a model of the conceptual organization of experience of the child, but from conceptual elements that are the researchers, and then trying for a "fit", which he distinguishes from a "match", with that of the child.

He maintains:

"The teaching experiment, as I suggested before, is, something more than the clinical interview. Whereas the clinical interview aims at

establishing "where the child is", the experiment aims at ways and means of "getting the child on"... In order to formulate any such hypothetical path, let alone implement it, the experimenter/teacher must not only have a model of the student's present conceptual structures but also an analytical model of the adult conceptualizations towards which his guidance is to lead."

(Page 62)

It is not clear if there is an operative difference between 'fit' and 'match', but this aside, the methods of clinical interview and the teaching experiment are proving interesting and potentially valuable research tools, and in this context, viable ways of identifying pupils' conceptual structures and paths of progression. As mentioned above, von Glasersfeld refers to Steffe's teaching experiments, and his discussion of clinical interviews is with reference to the work of Jere Confrey. Some recently published work of these two will now be described.

5.2.1 Steffe's 'teaching experiment'

The work of Steffe and his colleagues has been particularly in the area of young children's counting strategies and methods. It is fully described in a recent book (Steffe 1983) and reviewed by Carpenter (1985) as mentioned above. In a recent article (Cobb 1983), Cobb and Steffe describe their approach and method.

They write in the summary of the article:

"The constructivist teaching experiment is used in formulating explanations of children's mathematical behaviour. Essentially, a teaching experiment consists of a series of teaching episodes and individual interviews that covers an extended period of time - anywhere from 6 weeks to 2 years. The explanations we formulate consist of models - constellations of theoretical constructs - that represent our understanding of children's mathematical realities. However, the models must be distinguished from what might go on in children's heads." (Page 83)

They characterise the similarities between constructivist and non-constructivist teaching experiments as that both are long-term studies, both are concerned with the change from one state of knowledge to another and in particular how children do it, and both are generally working with qualitative data. The difference, they maintain, is their view that:

"... it is not the adult's interventions per se that influence children's constructions, but the children's experiences of these interventions as interpreted in terms of their own conceptual structures. In other words, the adult cannot cause the child to have experience qua experience. Further, as the construction of knowledge is based on experience, the adult cannot cause the child to construct knowledge. In a very real sense, children determine not only how but also what mathematics they construct. Consequently, we do not attempt to study children's construction of preselected processes in instructional contexts. Instead, we attempt to understand the constructions children make while interacting with us." (Page 88)

They go on to give an example of some work with a six-year-old boy, Jason. Their description highlights the essence of their teaching experiment method, in that they have developed a conceptual model of the stages of knowledge growth in counting, but their emphasis is on analysing Jason's conceptual understanding and giving him new tasks to extend his experience. They then use his progress to further elaborate their model of counting types. They end with an interesting note that teachers, when given the researcher's model of counting types, will construct their own understanding of children's mathematical realities, and research in how to help teachers is of "critical importance" (Page 93). The authors claim also, that their method is consistent with Lakatos' methodology of scientific research programmes (Page 92).

5.2.2 'Clinical Interviews' as used by Jere Confrey

In two recent articles, (1983(a), 1983(b)), Confrey describes some work done educating mathematics teachers with a view to their re-evaluating their teaching styles and philosophies, and also a course for young women in high school who have experienced difficulties with mathematics. In both, the clinical interview plays a major part in her research on students' mathematical thinking. Elsewhere (1981(a), 1981(b) pages 128-129) Confrey describes in detail her perception of the clinical interview, its value as an educational tool, and proposes directions for the development of the method.

She describes the usual interpretation of the clinical interview as (Confrey 1981(b)):

"... task-oriented, flexible interviews between a student and interviewer wherein the interviewer is expected to follow and pursue the student's thinking, asking questions until the student's reasons for response are understandable to the interviewer." (Page 6)

She expresses her dissatisfaction with these aims, and her criticisms include the need for "dynamic intervention" (page 7), and the need to "know not only how the student got the answer but also why the answer was given" (page 8). This latter concern stems from Confrey's view of the growth of knowledge as influenced by Toulmin, Quine and Lakatos (pages 9-14).

Her own view of the clinical interview is thus:

"A clinical interview aims to examine students' understandings of propositional knowledge, concepts, processes and reasons for believing in those concepts and processes. It can be based on a change perspective through which the interviewer attempts to ascertain what a student believes, why s/he believes, how s/he came to believe it and what predictions s/he might make as a result of

those beliefs. Both the interviewee and the interviewer assume active roles in the process, with the student for the most part guiding the inquiry. At times, the interviewer strives to clarify the meaning of the interviewee's statements, while at other times, s/he is more interactive, actively hypothesizing about the implications of the student's responses, posing new questions to test those hypotheses." (Pages 14-15)

5.3 Summary

Without entering into a comparison of the merits of clinical interviews as against the teaching experiment, it seems clear that research from the constructivist perspective is proving very valuable in providing information and insights of how children construct their knowledge, with the role of the teacher integrally involved. Whereas much research on mathematical thinking is descriptive, and the task of the teacher to intervene and assist children's conceptual growth at least unclear, this approach has the interventions of the researcher built in to the work. For Steffe and his associates, it is clear that the researcher must be in the role of teacher (Cobb 1983). Similarly, Confrey has emphasised the need for the researcher to understand "the influence which the context of classroom instruction has on ... performances" (Confrey 1982, page 27)

In this chapter I have attempted to survey some of the recent developments in research from a cognitive psychology perspective of mathematics learning and teaching. The links of these new directions with the philosophical view outlined in this thesis as the relativist/Lakatosian position are, I suggest, strong, and the elaboration of the alternatives of epistemological positions and the consequences outlined in this thesis provides, I propose, a much needed philosophical background and context, to this research.

SECTION 3

**A STUDY OF TEACHERS' ATTITUDES
AND WAYS OF TEACHING**

CHAPTER 6 - THE DEVELOPMENT OF THE STUDY

6.1 Background and Rationale of the Study

In this thesis thus far, I have attempted to establish a strong resonance in the reader for the position that there is a significant correspondence between the alternative philosophical views and different ways of teaching described in Section 2. The two stages of the study form an attempt to examine some of the consequences of this correspondence.

Attempts to examine these ideas empirically lead immediately to several research questions:

- (a) Since teachers' views are likely to be 'scarcely coherent', is it possible to determine their philosophical position in relation to mathematics?
- (b) Is it feasible to identify a tendency towards a particular way of teaching of any individual teacher?
- (c) Will any attempt to investigate (a) and (b) result in a clear correspondence between views and teaching, and if not, will it be possible to attribute this to any identifiable causes?

Despite these difficulties, it is proposed here that the contribution to the theory of mathematics education developed in this thesis has direct implications in mathematics education in general and in what goes on in the classroom in particular. Consequently it is important to begin to address these issues, and this is the aim of the study. At the very least, any such study will indicate some areas where further research is valuable.

The development of the first stage of the study is discussed in the remainder of this chapter. In Chapter 7 the results are presented and

discussed, from which arises the programme for the second stage of the study. Chapter 8 describes and discusses the results of stage 2.

Thus the questions for this first stage are:

- (i) can one design an instrument that will examine and identify teachers' views of the nature of mathematics and mathematics education?
- (ii) can one determine a criterion of 'open' teaching that is observable in a mathematics lesson, and find a suitable instrument of classroom observation to focus on that criterion?
- (iii) what is the nature of the match between (i) and (ii)?

6.2 Methodology

The methodology adopted for this first stage study is described in detail below, in sections 6.3 and 6.4, but, in outline, it was as follows:

(i) It was decided to develop a questionnaire, as the most appropriate method of examining teachers' views. It was felt that completing a suitable questionnaire would be less pressured than a direct interview, considering the difficulty of the area. With thorough planning and pilot field work of the questions, and careful analysis of the responses, the questionnaire could prove a suitable instrument for this purpose. It was also decided that a short interview with each teacher, to allow them to comment on the questionnaire, to record information on the teacher's background, training and experience, and to establish a personal rapport before observing lessons, would be appropriate.

(ii) An aspect of teaching method that is at least a necessary component of 'open' teaching, is the type of questions the teacher asks, the type of pupil interventions that are encouraged and elicited by the teacher, and the way he or she responds to novel interventions from pupils. This was adopted as the focus for observation. It also decided to adopt one

of the observation schemes that have been developed, rather than attempt to devise a new one, at least at the initial stage of the study.

It was decided to carry out a study initially in a single school, with all the teachers in the mathematics department who were prepared to participate.

6.3 Method of Examination of Teachers' Views

6.3.1 Questionnaire Development

Post, Ward and Willson (1977) recorded results of a study aimed at observing differences in views about mathematics teaching between teachers, principals and university faculty. They used a questionnaire, the Mathematics Inventory for Teachers assembled by Bracht (1972), which was given to teachers to answer directly, and to principals and university mathematics educators to answer:

"...as they believed an ideal mathematics teacher would." (Page 332)

The results and conclusions are interesting and bear on this study in one aspect in particular, which will be discussed in more detail in the review of this first stage study, that of the constraints on a teacher by virtue of working within a wider organization.

The use of a questionnaire for the purpose of examining teachers' views seems most appropriate and hence it was decided to develop a questionnaire for this study. The Bracht questionnaire was considered not ideally suitable for this study for the following reasons:

- (a) the statements appeared to be more relevant to North American schools,
- (b) there was not enough orientation towards determining attitudes to the nature of mathematics itself,
- (c) arising out of the situation in North American schools in the 1970's the study was partly directed at identifying cognitive as against

behaviourist objectives in teachers' views. This emphasis is not considered relevant as a focal point of this study.

It was decided to use a Likert scale, as in the above study, but modified by the addition of a "no view" option, as in informal pre-testing of the questionnaire a number of the teachers put no mark rather than be forced into an opinion. There are clearly advantages and disadvantages of including this option, and this will be discussed in the results and conclusions of the first stage study, in Chapter 7.

Thus, a questionnaire consisting of 35 statements was developed, and refined through informal testing and discussion with individual teachers, colleagues and supervisors. Initially, a group of constructs of aspects of secondary mathematics education was drawn up, under the headings of mathematics education, and mathematics, that involved teachers making principle decisions, either overtly or otherwise, about their teaching views and about mathematics. These constructs are set out in table 6.3.1(a)

Some of the statements designed to examine these constructs proved to contain more than one idea that required comment. For example, the first draft of the questionnaire contained a statement relating to syllabus content, as follows:

"Public examination syllabus, style etc. should be decided by experts not actively involved with teaching, e.g. mathematics professors, mathematicians in industry etc., but with consultation with teachers."

Clearly statements like these required considerable revision, in order that they would be unambiguous, and that one could estimate likely teacher responses.

Table 6.3.1(a) Questionnaire Constructs

(a) mathematics education:

- i who determines syllabus content
- ii active/passive learning, creative/non-creative learning
- iii aims of mathematics teaching
- iv the esoteric nature of mathematical knowledge
- v purpose in learning mathematics
- vi attitudes to discovery learning at secondary level
- vii alternative models - acquiring skills to be applied later
- developing ways of thought appropriate
to all mathematical work
- viii attitudes to open-ended questions, investigations etc.
- ix problem-solving skills

(b) mathematics:

- i role of proof in mathematics
 - ii the certainty or otherwise of mathematical knowledge
-

It was also decided to mix the statements, rather than keeping together things that related to the same construct. For example, in dealing with (a) ii, whether learning is seen as an active process on the part of the learner, or a passive one, a creative one or otherwise, five statements were to be used:

- "5. Learning mathematics in school is essentially an active process.
- 6. Learning mathematics in school is mainly a passive process.
- 7. Creative work in mathematics only takes place at the frontiers of mathematical knowledge.
- 8. Creative work in mathematics takes place in all learning.
- 9. Expecting pupils to be creative in mathematics is unreasonable and doesn't warrant use of precious lesson time."

It became clear that this arrangement encouraged people to make a choice between alternatives, whereas they might wish to answer "agree" to more than one statement relating to a particular aspect of mathematics teaching, e.g. 5 and 6 above.

The third change was in deciding to ask teachers filling in the questionnaire to make an ordering of importance of aims of mathematics education, rather than further agree/disagree marks. It seemed more appropriate to reveal teachers' views by asking which aim was a priority and which least important. Otherwise it was highly likely that all teachers would agree with all the aims, even though some they considered less important than others.

Based on the arguments of this thesis, a method of assessment of answers to the questionnaire was developed. Answers were to be given a numerical value according to their correspondence to the ends of the continuum described in Chapter 4. The scale of marking is from 1 to 4, with 2½ assigned to "no view" statements. Possible criticisms of the designation procedure are discussed in Chapter 7. Some statements were not assigned numerical values in this first stage study. This was because they appeared to be not sufficiently discriminating, but were retained for completeness. The examination of all the items for their discriminatory facility was developed in the second-stage study. The items not assigned marks in the final questionnaire were items 1,2,8,16,20. As a result, the range of marks was from 30 to 120,

although it was recognised that in practice few teachers commit themselves to "strongly agree" or "strongly disagree" very often, perhaps due in part to an awareness or self-consciousness of a university-based researcher asking the questions. Hence in practice the expected range is closer to 60 to 90.

Table 6.3.1(b) is a breakdown of the items of the final version of the questionnaire, into the two alternative views, called here 'absolutist' and 'fallibilist', assuming an "agree" response was given:

Table 6.3.1(b) Items as 'Fallibilist' and 'Absolutist'

	<u>Fallibilist</u>	<u>Absolutist</u>
Section (a)	2, 7, 10, 11, 12, 14, 15, 23	4, 5, 6, 8, 9, 16, 18, 19, 21, 24
Section (b)	31, 33, 34	29, 30, 32, 35

Table 6.3.1(c) is a breakdown of the items according to the constructs. The ordered pairs after each construct indicate the number on the first draft of the questionnaire as the first term of the pair, and the item number on the final draft as the second term of the ordered pair. Where a blank occurs, the final item does not correspond to any item on the first draft.

Table 6.3.1(c) Items by Constructs

	<u>Items</u>
(a) Mathematics Education	
i who determines syllabus content	(1,1) (2,7) (3,13) (4,16)
ii active/passive learning, creative	(5,2) (6,8) (7,9) (8,12)
non-creative learning	(9,17)
iii aims of mathematics teaching	(10,25) (11,26) (12,27)
	(13,28)
iv esoteric nature of mathematical	(14,3)
knowledge	
v purpose in learning mathematics	(15,18) (16,19) (17,20)
	(-,21)
vi attitudes to discovery learning at	(18,4) (19,10) (-,14)
secondary level	
vii alternative models - acquiring skills	(20,5)
- developing ways	(21,11)
of thought	
viii attitudes to open-ended questions etc.	(25,22) (26,23) (27,24)
ix problem-solving skills	(22,6) (26,15)
(b) Mathematics	
i role of proof in mathematics	(30,29) (31,33)
ii the certainty or otherwise of	(32,30) (33,31) (35,32)
mathematical knowledge	(36,34) (34,35)

Note: Each ordered pair represents, as the first term, the number of the item in the first draft, and as the second term the number of the item in the final questionnaire.

The first draft of the questionnaire, the final form of the questionnaire as used in the two stages of the study, with the columns for teacher responses, and also the marking scheme, form Appendix A. The items as they appear in the final form are listed in table 6.3.1(d).

Table 6.3.1(d) The Final Questionnaire

Section 1 - Mathematics Education

- 1) The examination syllabus largely determines the school syllabus.
- 2) Learning mathematics is essentially an active process.
- 3) It is a consequence of the nature of mathematics itself, that pupils will more often wonder about the purpose of a topic in mathematics than in, say, geography.
- 4) Discovery methods of learning mathematics are relevant for the earliest concepts only, e.g. addition, volume etc.
- 5) School mathematics can be seen to provide the basic skills and techniques of mathematics, to be extended into applicable mathematics in work or college situations.
- 6) The major value of teaching problem solving skills is to enable pupils to tackle unusual exam questions.
- 7) The examination syllabus should not be the main factor in determining the school syllabus.
- 8) Learning mathematics in school is mainly a passive process.
- 9) Creative work in mathematics only takes place at the frontiers of mathematical knowledge.
- 10) Discovery methods would be useful for older pupils if time and syllabus permitted.
- 11) The process of doing mathematics in school can be seen to be a model of all mathematical experiences: industry, research, daily life etc.
- 12) Creative work in mathematics takes place in all learning.
- 13) Teachers should be able to influence public examination boards in the style of examination, syllabus content etc.

14) Discovery learning of mathematics is relevant for all stages of school mathematics.

15) The development of problem-solving skills in pupils should be seen as an essential part of school mathematics.

16) Public examination syllabus, style, etc. should be decided by experts.

17) Expecting pupils to be creative in mathematics is unreasonable and doesn't warrant use of precious lesson time.

If an intelligent pupil were to ask the purpose of a topic in mathematics, I would answer:

18) Mathematics can be seen to be, like chess, a game with rules that have to be learnt.

19) The applications of mathematics follow once mathematical knowledge is acquired.

20) Mathematics is training you to be logical.

21) This topic is on the syllabus, so you can rely on its importance.

In the following statements, the term 'open' is used to describe questions such as "how could we add $\frac{1}{2}$ and $\frac{1}{3}$ " to pupils who have not learnt the algorithm, since the question could be answered in many ways.

22) Asking open questions is essentially just a useful device in teaching.

23) Asking open questions is vitally important as it gives pupils the opportunity for creative thought.

24) There are no open questions since both teacher and pupils know that there is always only one correct way to solve any problem.

Please give an order, 1 to 4, to the following aims:

25) My main aim is to try to enable every pupil to leave school with some public examination success in mathematics, GCE, CSE etc.

26) My main aim is to try to enable every pupil to become a mathematician to their level of ability, i.e. to be able to think mathematically where applicable.

27) My main aim is that school mathematics should be seen by pupils (and parents, employers, etc.) to be relevant and applicable to the real world.

28) My main aim is to enable pupils to appreciate and enjoy mathematics for its own sake.

Section 2 - Mathematics

- 29) Once a mathematical structure has been developed, and a theorem formulated, its proof is a technical detail, although it may be years till its discovery.
 - 30) Mathematical truths are not susceptible to revolutionary change in the way that scientific truths are, e.g. relativity.
 - 31) Mathematical knowledge is close to scientific knowledge in the sense that conclusions are tested for their truth.
 - 32) Mathematical truths have an inevitability about them. A world with different mathematical truths is inconceivable.
 - 33) The generation of a proof is a highly creative part of mathematics since it can lead to new structures, reformulated hypotheses, etc.
 - 34) Mathematical knowledge is hypothetical and potentially subject to refutation or adaptation.
 - 35) Mathematics is essentially hierarchical and cumulative. Although progress does go on making earlier work more rigorous, generally new knowledge builds on former work.
-

6.3.2 Validation of Marking Scheme

The marking scheme was tested in two ways:

- (a) For corroboration of the numerical values and their place in the continuum, the questionnaire was given to a teacher who had not been involved at all in the development of the questionnaire, but whose views were assessed to be close to the 'open' end of the continuum. He was asked to answer in that vein, and he scored close to the maximum, a high degree of corroboration.
- (b) For further corroboration and also to test for the questionnaire's ability to distinguish between teachers' views, rather than most gaining a similar mark, the questionnaire was given to six teachers in the school in which the above-mentioned teacher was Head of

Department. He was asked to order the teachers according to the continuum, and his knowledge of the teachers, and this ordering was compared to their marks on the questionnaire. The results are given in table 6.3.2.

Table 6.3.2 Correspondence of Marks & H.O.D.'s Predictions

<u>Head of Department's Ordering</u>	<u>Marks</u>
A	88
B	80½
C	73
D	77
E	63
F	58

As a result of these pilot tests, the questionnaire was adopted as an instrument for the purposes of the study. Further weaknesses, if any, would be revealed in the analysis of the results of the questionnaire from the teachers in the main study, and in conversations with the teachers in that school.

The questionnaire was given out to the teachers at a short lunchtime meeting, which was also the first time that there had been any contact between the researcher and the teachers, although there had been two discussions with the Head of Department on previous occasions to obtain agreement from the department to carry out the study. After a brief introduction, the research was described as being into teachers' views about teaching mathematics and, through observation, into different ways of teaching. The teachers were then asked them to take the questionnaire in a philosophical frame of mind, i.e. with answers that

reflected their beliefs about mathematics education, rather than just the character of the particular school in which they were teaching.

6.3.3 Interview

In order that the mathematical and educational background of the teachers could be recorded, and in order to establish a rapport with the teachers in the main study, it was decided to conduct short interviews with each teacher, after completion and marking of the questionnaire. This also gave opportunity for some discussion, or at least clarification, of apparent contradictions in answers. In an attempt to avoid influencing teachers' views with revelation of the views of the researcher, it was decided to avoid discussion in the interviews. The interview was also used as an opportunity to emphasise that no judgements were being made as to 'good' or 'bad' teaching, only attempts to identify different ways of teaching.

6.4 Method of Observation of Teaching

Before either choosing an existing observation tool, or developing a new one, it was clearly necessary to determine some focus of attention in pupil/teacher interaction, and a suitable classroom context, which in some fairly reliable way revealed something of different approaches to teaching, again on the hypothesised continuum of teaching outlined in Chapter 4. This necessarily meant making some choices, with the inevitable restrictions and drawbacks this implied.

As a classroom context, it was decided to choose a situation where most, if not all lessons were taught by a class teacher dealing with the whole class. The major reason for this choice is that observing lessons where the emphasis is on individualized learning is very problematic, from the point of view of analysing the kinds of teacher/pupil interaction. It involves following the teacher around and listening in to individual conversations in an intrusive way, or setting up a video system and recording lessons. It was felt that the less intrusions into the

classroom the better, as far as attempting to identify different ways of teaching is concerned.

The classroom incident discussed in 4.3.1 in which the two alternative teacher strategies in response to the unexpected pupil reply to the question of finding a fraction between two others, led to the idea that teachers' reactions to pupils' answers, and in particular to novel or challenging ones, and also teacher willingness to stimulate and encourage independent thought, are at least necessary indicators of an 'open' way of teaching, if not sufficient. Hence it was decided to focus on the occurrence or otherwise of these interactions.

It has to be recognised that observations of these aspects of classroom activity involves relatively high inference judgements, with the consequent limitations of the impossibility of replaying or verifying the subjective judgements of the observer. Nevertheless, as a first approach at researching these issues, it was decided to proceed with this focus of attention for the study.

There are clearly restricting implications of the choice of classroom context and focus of attention for observation. A school that has not chosen an individualized programme of mathematics is making a particular statement, possibly, about its view of mathematics education. Also there are connections between the two aspects of the particular focus of attention, in that a teacher who regularly stimulates a class with challenging open-ended questions is quite likely to get, and to be able to deal with, novel pupil reactions. Nevertheless, it was assumed, for the purposes of this study, that a school with a sizeable mathematics department, would contain teachers with a cross-section of beliefs, and also a variety of approaches to ways of teaching. The validity of this assumption will be discussed in Chapter 7, with the results of the study.

An alternative approach to investigating teachers' views and their ways of teaching is the case study method, as described above in Section 3.3 in work being carried out in the USA. The emphasis in those programmes

as in the present one, is on qualitative information, but the major difference is that this study results out of an attempt to examine thoroughly and completely the alternatives of approaches to the nature of mathematics, and the nature of the possible connections with teacher actions. Consequently it seems appropriate at this stage to take a whole department in a school and look at the variations in views and ways of teaching at some depth rather than a small number in great depth. Again, this aspect will be discussed in Chapter 7.

The observation tool to be used would need to reveal the aspects of classroom interactions discussed above. The options appeared to be the Systematic Classroom Analysis Notation programme, SCAN, developed at Nottingham University (Beeby 1979), or the Interaction Analysis programme of the Five State Project used in the USA in the 1960's (Five State Project 1963). The SCAN programme is a well-developed, sophisticated and complex one that provides the observer with considerable information of the classroom interactions. The particular information relating to the focus of attention of this study would then need to be extracted. The latter programme is much less complex, and the researcher is immediately involved with decisions as to the type of teacher talk or pupil talk that is taking place. For this reason it was decided to use the Interaction Analysis at the initial stage of the study, and if the results of the whole study were such as to suggest further work using this approach, it might be appropriate to use SCAN for the greater amount of information it would provide surrounding the specific focus of attention.

The instrument, reproduced as Appendix B, is based on Flanders work (Flanders 1962). It uses 10 categories of behaviour, 1 to 6 for the teacher, 7 to 9 for pupils, and 0 for general chaos, organization or silent work.

Teacher categories 3 and 4 are described respectively as:

"Confronting, seeking: remarks, usually questions, which invite extensive participation: classifying information, a series of steps, a single step requiring selection and organization of material."

and

"Soft or hard challenging, jolting: remarks absurd, controversial or questions of comprehensive nature or completely undirected to invite significant participation: in noting relationships, application, in making grand leaps in system development."

Consequently, tally marks, the method of recording incidents, in these categories are the centre of interest of the study as far as teacher talk is concerned. These are taken every 3 seconds.

The pupil categories 8 and 9 are similarly revealing, being:

"Independent, active: remarks by student either as invited and moving more than one step ahead, or a single powerful step, or without invitation, to raise a question and being willing to treat it himself."

and

"Curious, creative: remarks by student in which present topic related to other areas of mathematics or to applied fields, to more fundamental concepts, or to a wider family of topics. A fresh topic related to present topic."

The tally marks, which are taken every three seconds, are ordered pairs of numbers identifying the categories describing the classroom interactions are recorded in the lesson, and afterwards transferred onto a 10 by 10 matrix. The style of lesson is immediately obvious, in that markings in categories 3,4,8 and 9 can be seen in contrast to the markings in other categories.

Skill in using the tool was developed on video recordings of mathematics lessons.

6.5 School Selected

A school was chosen for the study, which commenced in January 1984, with the usual confidentiality assured and agreed at the beginning. The mathematics department consisted of 7 full-time mathematics teachers, one teacher who was full-time in the school but gave half-time to the mathematics department, and 4 part-time teachers. The mathematics course used is geared to an 'O' level specific to that school, and described by the Head of Department as a mixture of modern and traditional mathematics. None of the teachers was known to the researcher.

Each teacher was asked to provide up to three classes for observation, and which the teacher felt comfortable teaching, since discipline problems would be an interference to the observation aims. Also, the individual approach to teaching would be revealed more easily in situations in which the teacher felt more relaxed, rather than under stress to maintain control. This inevitably would affect the results, perhaps mostly in the sense that teachers might feel more able to invite and cope with challenging and stimulating interactions in classes where control is not a problem. This will be considered further with the results of the study.

Five lessons of each class were to be observed, in order to build up a picture of ways of teaching which balanced out over time in lessons for quiet work, revision, going over homework, etc.

Nine teachers agreed to participate in the study, a total of 16 classes were observed, and 80 lessons in all.

The results of the study, and a discussion of the method used, the research tools, possible explanations of the results etc. follow in Chapter 7.

CHAPTER 7 - RESULTS OF THE STUDY

The results of this first stage study are presented here in Section 7.1. First, two tables are presented, the first summarising details of the teachers in the study, and the second showing the questionnaire results within the constructs as outlined in Table 6.3.1(a). This is followed by a short profile resulting largely from the brief interview, the questionnaire results, and specific comments from the observation instrument, with respect to each of the nine individual teachers, assigned letters A to I, will be presented, in sections 7.1.1 to 7.1.9, whilst the full mark sheet of the questionnaire, and the tally sheets from the classroom observations, form Appendix C.

A full analysis of the implications of the results will follow in Section 7.2.

7.1 Results

The categories of Interaction Analysis that form the focus of the classroom observation are:

"Teacher Categories:

3. Confronting, seeking: remarks, usually questions, which invite extensive participation: classifying information, a series of steps, a single step requiring selection and organization of material.

4. Soft or hard challenging, jolting: remarks absurd, controversial or questions of comprehensive nature or completely undirected to invite significant participation: in noting relationships, application, in making grand leaps in system development.

Pupil Categories:

8. Independent, active: remarks by student either as invited and moving more than one step ahead, or by a single powerful step, or without invitation to raise a question and being willing to treat it himself.

9. Curious, creative: remarks by student in which present topic related to other areas of mathematics or to applied fields, to more fundamental concepts, or to a wider family of topics. A fresh topic related to present topic."

Table 7.1(a) The Teachers

Teacher	Age	Years of Experience Teaching Mathematics	Training	Questionnaire Mark	No. of Diff. Classes Obs.
A	34	13 (inc. 6 as H.O.D.)	BSc Maths PGCE	86½	2
B	42	1	BSc Science PGCE	77½	3
C	46	4 full, 6 part-time	MA Maths PGCE	83½	1
D	27	5	BSc Maths PGCE	77	2
E	36	4 part-time	BSc Maths	65	1
F	28	6	BSc Maths PGCE	82½	1
G	50	3 part-time	BSc Sociology	73	1
H	38	3	BSc Metallurgy PGCE (primary)	85½	3
I	35	2	BSc Chemistry PGCE	79½	2

Note: Teacher G had taught sociology full-time for several years, and was now teaching that part-time. Teacher H had taught in primary schools for 7 years. Teacher I had taught Chemistry for 8 years.

Table 7.1(b) Questionnaire scores within constructs

Construct	Teachers									
	A	B	C	D	E	F	G	H	I	
(a) i who determines syllabus content	6	5	5	6	4	8	5%	5%	6	
ii active/passive learning, creative/non-creative learning	8	8%	10	8	8	6%	6%	10	7	
iii aims of mathematics teaching	14%	8%	10	12	8	11	5%	11	13	
iv the esoteric nature of mathematical knowledge	2	2	2	2	2	3	1	2%	2	
v purpose in learning mathematics	8	9	7	7	6%	8	9	8	7	
vi attitudes to discovery learning at secondary level	9	9	9%	7%	6%	11	7%	11	9	
vii alternative models-acquiring skills to be applied later	2	2%	3	2	2	2	2	1	2	
-developing ways of thought appropriate to all mathematical work	3	2	3	2	2	3	2	2%	3	
viii attitudes to open-ended questions, investigations etc.	8%	8	8	10	7	11	10	11	9	
ix problem-solving skills	6	4%	6	7	4	7	6	6	5	
<hr/>										
sub-total of (a)	67	59	63%	63%	50	70%	55	68%	63	
<hr/>										
(b) i role of proof in mathematics	5	5%	6	5%	4	6	5%	5	5	
ii the certainty or otherwise of mathematical knowledge	14%	13	14	8	11	6	12%	12	11%	
<hr/>										
sub-total of (b)	19%	18%	20	13%	15	12	18	17	16%	
<hr/>										
Totals	86%	77%	83%	77	65	82%	73	85%	79%	
<hr/>										
Rank	1	6	3	7	9	4	8	2	5	

A few remarks only will be made in relation to the breakdown of teacher scores within constructs. In other circumstances, it is proposed, a full analysis of this table could be carried out. However, there are a number of aspects that limit the value of this exercise at this stage, with these particular results:

(a) if one is going to carry out this exercise, it is perhaps more appropriate to first examine the character of the items in the questionnaire for their ability to serve the purpose required, before the above analysis, since if a number of items are not in the most suitable form, the detailed analysis would be invalid. This examination of the items would require a larger sample than the group of teachers here, and forms one of the possibilities for the second-stage study.

(b) the intention at this stage is to gain an overall picture of the views of teachers, particularly in some sort of ordering, rather than a detailed determination of views.

Some of the constructs have a number of items relating to them, whilst others have only one or two. Clearly these former will thus contribute a large amount to the overall score. It might be desirable to balance out this difference, but there is a danger of the size of the questionnaire becoming too unwieldy. There is however only a weak match between the ranking of the overall scores and a ranking of any of the large contributors of marks. These are (b)ii with five items, (a)ii and iii with four items each, and (a) v, vi and viii with three items each. This may be due to what Thom termed teachers' 'scarcely coherent' philosophy of mathematics (Thom 1973 page 204). It might equally be due other factors, such as the teachers completing the questionnaire in a short space of time, under some pressure etc. Similarly, there is only a weak match between the ranking of the two separate sections of constructs, (a) and (b). Indeed, in one case, teacher F actually scores highest in section (a) and lowest in section (b).

These results might indicate a need for further work on the questionnaire, and this is discussed at the end of this chapter.

7.1.1 Teacher A

Teacher A, the Head of Department, was a mathematics graduate, had taken a Postgraduate Certificate in Education, and had taught for 13 years, including six at this school as Head of Department.

He considered himself a traditionalist with regard to mathematics, choosing for example an 'O' level syllabus that treated geometry in a "Euclidean manner", as he did "not personally like matrices". He was a strong disciplinarian with school pupils. He was fairly firm and rigid with his department (the researcher's observation), setting out syllabus and targets for each class, choosing textbooks for the department, and comparing test and examination marks with the same sets in previous years. He said that he called meetings of the mathematics department staff when necessary, which he said generally meant when there was administration to be dealt with. In brief talks with the headmaster, it was clear that Teacher A had full support in his approach from the headmaster, who on a number of occasions emphasised the success of the department. The school overall had good examination results in mathematics and generally in other subjects, compared to other secondary schools in the area, and is considered by local parents to be one of the most popular, and 'best' schools in the area.

With regard to the questionnaire, Teacher A said he recognised "certain standard items which demanded standard answers" but that he had "tried to answer honestly".

The teacher offered two classes to be observed, a top stream set in the fourth year, preparing for an early 'O' level at the end of the fourth year, and a bottom set in the fifth year.

Teacher A scored 86% marks on the questionnaire, the highest total of all the teachers in the study. The answers exhibited a number of contradictory views, for example he agreed with item 29 regarding proofs in mathematics as technical details once a theorem has been formulated, and also agreeing with item 33 regarding proofs as a highly creative

part of mathematics. He agreed with both constructs of alternative models for learning mathematics, as acquiring skills to be applied later and as developing ways of thought appropriate to all mathematical work. In the interview, when asked if he wanted to discuss the answers given to the questionnaire, he declined.

As regards the lesson observations, in the Fourth Year set, on only two occasions throughout the lessons observed did the teacher ask a question other than a category 2 question. These were both of category 3, and neither were responded to by pupils at other than category 7. There is a concentration on the 'lecturing' element of the tally sheet matrix, (5,5) and a heavy concentration on the sequences (1,2) (2,7) (7,1), which represents a cycle of: recognition of pupil response, onto a simple question, receiving a simple reply, and acknowledging that reply. The only other elements with a large number of entries are those that transfer to or from the lecturing element.

In the Fifth Year set, the teacher avoided almost all talk, choosing to lecture briefly, (5,5), ask only one or two simple questions, and then set class work, with little or no summary talk at the end. It seems likely that this pattern was adopted because of potential behaviour problems, although in all the lessons observed the teacher clearly had the class completely under his control. There were no entries in the matrix for category 3 or 4 questions or 8 or 9 pupil responses.

General comments about the match or otherwise between teachers' results on the questionnaire and their ways of teaching will be made below, but it is perhaps significant in this particular instance to note that the teacher's comment on "standard questions" may have been part of the cause of his mark being the highest of all the teachers in the sample. Certainly his teaching did not match his responses to some of items on the questionnaire, such as his support for 'open' questions in items 22 and 24.

7.1.2 Teacher B

Teacher B was a mature man in his probationary year of teaching. He had originally taken a combined honours degree in Physics and Chemistry, and then worked for many years as a maintenance engineer in industry. He had decided to change career to teaching, and had taken a full-time Postgraduate Certificate in Education in secondary mathematics. He emphasised in his interview that because of his own work experiences, he wanted "pupils to be able to use their mathematics", and as such his main concern was "to try to make" his "pupils understand mathematics, with the emphasis on understanding".

He offered three classes for observation, two first year classes, one in the top ability band and the other in the bottom ability band, and a second year bottom ability set.

Teacher B scored 77% on the questionnaire. At the end of the questionnaire, he wrote:

"In my view very few pupils are natural mathematicians therefore it is essential to provide more links between allied scientific subjects and real life situations. This is particularly relevant to less able pupils."

In the interview, his only further comment on the questionnaire was that he had found section 2 "very difficult". This is reflected in his "no view" response to 3 of the seven questions. Surprisingly perhaps, given his expressed opinion, he had a "no view" response to item 5, which suggests that school mathematics provides the skills to be applied later, and a "disagree" response to the other related item, no. 11, that school mathematics is a model of all mathematical work.

As may be seen from the tally sheets, in his two bottom band sets, he did not ask questions of any category other than 2, nor did he receive any pupil responses at any category other than 7. With the first year bottom set there was a considerable amount of teacher and teacher/pupil

talk, of either lecturing (5,5) or short questions and answers (1,2) (2,7) and (7,1). In his first year top band set, he asked 3 questions of category 3, but received replies from the class at category 7. There was a considerable amount of teacher talk and teacher/pupil talk, as with the other set, and in the same manner. The second year bottom set spent most of the lesson working alone, there was much less teacher talk, due probably to fear of indiscipline. In the interview, the teacher had commented that he allows "quite a high level of noise" in his lessons, so long as there is work going on.

7.1.3 Teacher C

Teacher C read mathematics at Cambridge, took a postgraduate Certificate in Education, and then taught for four years at a girls' private school. She stopped teaching for a number of years, for family reasons, and had then returned to part-time teaching. She was in her seventh year at this school.

This teacher offered just one class for observation, a top third year set, heading for 'O' level at the end of the fourth year.

Teacher C scored 83% on the questionnaire. She did not wish to comment further on the questionnaire, in the interview. There are few apparent contradictions in her answers. The only notable one is in disagreeing with both items 9 and 12, the first of which suggests that creative work only takes place at the frontiers of knowledge, and the second which states that creative work takes place in all mathematics learning. In her interview, she commented that she has a very "pure mathematics view of the subject" due probably to her particular university experience, and also her private school teaching, she said.

As can be seen from the tally sheet of the class observed, there is a considerable amount of teacher talk, and teacher/pupil dialogue. Whilst she often set the class some work, it was for a short time on each occasion, and they were quickly brought back together. This teacher asked a number of questions at level 3 and some responses from the

pupils were at level 8. There were also a large number of simple questions and answers, as seen again from the concentration of marks in (1,2) (2,7) and (7,1).

7.1.4 Teacher D

Teacher D had graduated in mathematics, obtained a Postgraduate Certificate in Education, and was now in her fifth year of teaching, all at this school.

She offered two classes for observation, the top set in the Fifth Year, who had all taken 'O' level the previous June and were now preparing for an Alternative Ordinary level in Pure Mathematics, and a Second Year class, which was fourth set in ability, out of eight sets.

Teacher D scored 77 in her questionnaire. She scored fairly low in the second section, on mathematics. She also commented in her interview that she found that section very difficult. This was her only comment on the questionnaire. She was second in the department in responsibility. In the course of this study, it was revealed that a number of the teachers felt that they did not have much influence in the decisions on the mathematics curriculum and the style of teaching in the department. This teacher said that she would perhaps teach differently if she had more control over her own teaching. She also said that in her teaching she liked to ask lots of questions of the pupils, and that learning often took place best when they were first passive, and then proceeded by discovery.

In both classes observed, there was a great deal of teacher talk and teacher/pupil talk. In the Second Year class, there were a small number of questions of category 3, and some of these were responded to at level 8. In the fifth year set there was just one question at level 3, although there were three interventions from pupils at level 8. One of these was an attempt by a pupil to offer a solution to a new problem, to which the teacher replied that pupils should not attempt questions that they had never seen before.

At the end of the observation period, teacher D volunteered the information that she had been very hesitant at being observed, and that she had noted that she tended to talk much more than usual as a consequence of the presence of the researcher.

7.1.5 Teacher E

Teacher E graduated in mathematics, and then worked as a mathematician in a scientific research establishment for four years. After having a family, she began part-time teaching, and was now in her fourth year of this, but her first at the school of the study. She taught Sixth form and one low ability 1st Year class, this latter she offered for observation. She was very hesitant in the interview, and not very enthusiastic about being observed, but said she wanted to participate in the research.

She scored 65 on the questionnaire, expressing "no view" seven times, and no "strongly agree" or "strongly disagree" responses at all. In particular, she had "no view" on 4 of the seven items in the second section. She had no comments to make on the questionnaire, when interviewed, except to say that in secondary schools, where there is any discovery learning, it is guided discovery, which, she said, "is not quite the same thing".

Her lessons were almost completely taken up with trying to gain control of her class and she was obviously embarrassed at being observed in this. Her tally sheet is included here in the Appendix, but cannot be used for drawing any conclusions about her teaching, for this reason.

7.1.6 Teacher F

Teacher F graduated in mathematics, took a Postgraduate Certificate in Education, and had been teaching for 6 years full-time. He considered himself to be a Pure Mathematician, and as evidence he offered his great interest in chess and in computing, during his interview.

He offered just one class for observation, a Third Year class, set 3 of eight in ability. He said that he had some difficulty in controlling his classes, and this was the reason that he would only offer one class to be observed. He added that it was not the case that this class was particularly well-behaved, but he felt that they and he could "handle being watched, and anyway it will be interesting to see the way they react".

Teacher F scored 82% on the questionnaire. He commented in the interview that he had enjoyed completing the questionnaire, found the questions stimulating and difficult, and felt that he wanted to discuss most of them. With the initial policy for the interview in mind, that of not revealing the views of the researcher, no opinions were offered, but he was encouraged to comment as he wished. He said that apart from agreeing that proofs are a creative non-trivial part of mathematics, he sees mathematics as a certain body of indubitable knowledge. However, in relation to mathematics education, he felt more 'open' in the sense that pupils have to be involved in the process of learning, themselves. He saw many tensions and contradictions in his own views. He said that his main aim for lessons was to get the pupils to practice their acquired knowledge. He was more prepared to use the "strongly agree" and "strongly disagree" responses than any other teacher in the group.

As may be seen from his tally sheet, there were no entries in the 3 and 4 columns and rows, nor in the 8 and 9 columns and rows. His lessons were not unduly noisy or undisciplined. There were a large number of simple questions and replies, and also quite a lot of 'lecturing' recordings.

7.1.7 Teacher G

Teacher G graduated in sociology, but over a number of years of teaching had begun to teach more mathematics. She said that her own mathematical knowledge resulted only from her own 'A' level passes in separate Pure and Applied Mathematics. This meant that she felt very insecure in her teaching of mathematics, and thus nervous of being observed. She

suggested that through her sociological training she was very aware of the ethos of the school, which she described as being "traditional", and also of the framework set within the mathematics department. She added that she, in particular, due to her limited knowledge of mathematics, was very influenced by the programme set by the Head of Department. She also said that she had had no observation of her mathematics lessons before, which she regretted, but that this made her even more nervous of the researcher.

She offered just one class for observation, a Second Year, fifth set of eight.

Teacher G scored 73 on the questionnaire. She wrote, at the end of the questionnaire:

"I feel the ideas underlying this questionnaire do not come to grips with the actual day to day classroom situation."

In the interview she explained that she meant that the questions were clearly theoretical, whereas what actually happened in the classroom was quite different. She offered answers, rather than "no view", to most of the second section, and in nearly every case took a fallibilist view of mathematical knowledge. She did not, however, feel that learning mathematics is creative, nor did she feel that discovery learning is relevant for concepts other than the earliest in mathematics.

Her tally sheet shows an emphasis on 'lecturing' and on simple questions and answers. There was one question recorded at category 3, and no pupil responses other than category 7. She commented after the observation that she had tended to talk too much, out of nervousness.

7.1.8 Teacher H

Teacher H graduated in metallurgy. After having a family she took a Postgraduate Certificate in Education in primary teaching. She taught mathematics in a preparatory school, and had been in the school of the

first-stage study for 3 years, with responsibility for computing. This was taught at CSE and 'O' level in the 6th form only. The remainder of her time was taken up with teaching mathematics. In the interview, she said that in her view, mathematics was a tool of the sciences, and of the real world, and that this was her view of teaching.

She offered three sets for observation, a Second Year top set, a Third Year low ability set, sixth out of eight, and a Fifth Year set 2, preparing to take 'O' level later that year.

Teacher H scored 85% on the questionnaire. She offered "no view" for all the items relating to aims, numbers 25 to 28. She explained that it depended entirely on which level of ability one was teaching, and therefore she could not give a general answer. She said that she recognised that there were some contradictions in her answers, but said that this was due in part to her 'tools' view of mathematics, and in part to her strong religious views as far as truth was concerned.

In her Fifth Year set there was a particularly high concentration of marks in the lecturing element (5,5), although it is also quite high in both the other classes. There are also a large number of entries in the short simple question and answer categories. In the Second Year class there was one entry in the category 3 and one response in category 8. In the Fifth Year there were two entries in each of these, but none at all in the third year class.

7.1.9 Teacher I

Teacher I graduated in Chemistry, worked as a University librarian, and then taught Chemistry for 8 years, which she stated she found "very good and very valuable". She took a Postgraduate Certificate in Education in secondary mathematics, and was now in her second year of teaching at this school. She stated in the interview that her aim was to keep her pupils "switched on" in their learning. She considered mathematics as a tool of science, and said that "real mathematics is applied mathematics".

She offered 2 classes for observation, a First Year class in the top ability band, and a Second Year class, set 2 in ability. She expressed a certain degree of insecurity at being observed by a mathematician, but in fact she was quite confident in her lessons.

Teacher I scored 79% on her questionnaire. She commented that she was in favour of more discovery learning and open questions in class. There were some contradictory answers in the second section, in that, for example, she agreed both with 29, that proofs are technical details, and with 33 that they are highly creative parts of mathematics.

As may be seen from the tally sheets, teacher I asked a few questions in both classes at category 3, and received pupil responses in category 8. There were very high concentrations on 'lecturing' and on simple questions and answers.

7.2 Discussion and Possible Criticisms of the Study

As discussed earlier in Section 4.2. the extremes of the continuum of teaching method, 'closed' and 'open', are intended to highlight the hypothesised implications of the adoption of a particular epistemology of mathematical knowledge. It is not proposed here that any teacher will represent either of these extremes. The study shows how far one can answer the questions asked, with the kinds of data collected.

In 7.2.1 some general comments will be made about the results of the study. In 7.2.2 more specific comments will be made about the suitability of the research tools, the teacher/pupil interactions chosen as the focus of the study, etc. In 7.2.3 some conclusions are drawn, and possibilities for a second stage study discussed.

7.2.1 Some General Comments

The teachers in the study represented a fairly wide range of backgrounds and preparation for the teaching of mathematics, including graduates of mathematics, sociology, physics, chemistry and metallurgy. Most but not all had completed a Postgraduate Certificate in Education in Secondary Mathematics. Some had come straight into teaching after university, others had been in industry or other employment before teaching. There were teachers in their first years of teaching, and others with many years of experience in schools. One would, I suggest, have expected, or have good reason to expect, quite a wide variation in views about mathematics and mathematics education, and in ways of teaching.

The marks on the questionnaire range from 65 to 86%. When seen against the full range of possible marks 30 to 120 this represents a uniformity of views, but when one notes that only two teachers used the categories "strongly agree" or "strongly disagree" more than four times in total, the more realistic range of 60 to 90 becomes more appropriate here. Within this range, the spread of marks is quite wide. Neither the upper end, nor the lower end, were gained by either of the teachers who made wide use of the strongly held views categories. These latter were teachers F and H, whilst the maximum mark was obtained by teacher A and the minimum by teacher E.

As regards the Interaction Analysis, the picture is different. The tally sheets for each class may be seen as giving a general and fairly vivid picture of the kinds of classroom interaction taking place in any particular classroom. The density of the tally marks in any square of the matrix indicates the frequency of that category of teacher or pupil behaviour and one can soon gather an impression of the character of particular classes and teachers. Also, one can use the number of entries for more detailed numerically-based analysis, by comparing percentages of particular categories, or by counting the number of instances of incidents considered important. It was decided that the percentage work, carried out in the original Flanders work, is not valid here, for two reasons: firstly there is a predominance of (0,0) entries

in all the lessons, due presumably to British ways of teaching which include individual pupil work in almost every lesson, and sometimes quite a considerable amount of individual work; and secondly because the focus is on specific incidents, that is the occurrences, or lack of occurrences, of entries in columns and rows 3,4,8 and 9. This was the focus of attention for the study, that is, those categories that are hypothesised as identifying 'open' teaching, or at least indications of necessary conditions for 'open' teaching. The table shown below, 7.2.1, shows the number of tally marks in these categories, for each teacher, in each class observed, over the total of lessons.

Clearly, considering that these totals are for all five lessons of each class, they are very small in all cases except Teacher C. There were no marks for any teacher, or any pupil, in categories 4 or 9.

This suggests that there is a very uniform pattern of teacher/pupil interaction in this school, with these teachers, and during the particular lessons observed. The teachers, with the possible exception of Teacher C, do not, in general, ask any questions of any greater depth than immediate recall or simple deduction. The pupils, perhaps for reasons connected to the teaching method, do not respond to the few category 3 questions with an equal number of category 8 answers, i.e. they do not rise to the challenge of those few questions that are more extending than recall or simple questions. They also do not ask questions or make contributions to the lessons, at these levels.

Hence if the focus of attention for the study is valid as an indicator of teaching methods, there is a uniformity close to the 'closed' end of the continuum, despite a fairly wide range of marks showing the views of these teachers about the nature of mathematics and mathematics education. This uniformity is most evident in those categories which describe and identify more open-ended questions from teachers and novel ideas from pupils, categories 4 and 9 respectively, in which there are no marks at all, for any of the teachers, in any of the classes observed.

TABLE 7.2.1 Tally mark totals in Categories 3,4,8 and 9

		Category 3	Category 4	Category 8	Category 9
Teacher A	(1)	3	0	0	0
	(2)	0	0	0	0
Teacher B	(1)	0	0	0	0
	(2)	0	0	0	0
	(3)	3	0	0	0
Teacher C	(1)	13	0	7	0
Teacher D	(1)	5	0	3	0
	(2)	1	0	3	0
Teacher E	(1)	0	0	0	0
Teacher F	(1)	0	0	0	0
Teacher G	(1)	1	0	0	0
Teacher H	(1)	1	0	1	0
	(2)	0	0	0	0
	(3)	2	0	2	0
Teacher I	(1)	5	0	4	0
	(2)	3	0	1	0

It is clearly important to try to identify the possible causes of this uniformity. One can, I suggest, learn much about the influences and pressures on teachers, that significantly affect the ways that they teach, from such an analysis.

A strong candidate for explanation of this uniformity may be that the ethos of the school and the particular mathematics department may be the most significant factors in determining the way that teachers teach. This was mentioned by one teacher directly, Teacher G, and indirectly through the criticisms of the Head of Department by some of the teachers in the study. There is also some evidence in other research to support this. Thompson (1984), in discussing the results of her case studies of teachers' attitudes to mathematics, concludes:

"... teachers' beliefs, views, and preferences about mathematics and its teaching, regardless of whether they are consciously or unconsciously held, play a significant, albeit subtle, role in shaping the teachers' characteristic patterns of instructional behaviour... Teachers possess conceptions about teaching that are general and not specific to the teaching of mathematics. They also have conceptions about their students and the social and emotional make-up of their class. These conceptions appear to play a significant role in affecting instructional decisions and behaviour. For some teachers, these conceptions are likely to take precedence over other views and beliefs specific to the teaching of mathematics." (Pages 124-125)

Post et al (1977) report a similar conclusion:

"By virtue of working within a larger organization - in this case the school - constraints are placed on an individual teacher's decision-making power. This tends to result in an apparent within-school conformity of thought and idea." (Page 339)

The results of this present first-stage study seem to suggest an apparent within-school conformity of ways of teaching in spite of some quite wide variation in teachers' views. For example, item 11 stated "The process of doing mathematics in school can be seen to be a model of all mathematical experiences: industry, research, daily life etc." Four teachers agreed, four disagreed and one had no view. Thus the conformity has not shown itself in thought and idea, as suggested by Post, but in the ways of teaching.

7.2.2 The Research Tools

The focus of attention chosen for observation by Classroom Interaction, i.e. those categories that are proposed as identifying necessary components of 'open' teaching, is discussed first, followed by the questionnaire and the interview.

(a) Lesson Observation

It is undoubtedly difficult to identify when 'open' teaching is taking place, since the content or format of a lesson do not necessarily distinguish these ways of teaching by themselves. Investigative work, for instance, can be going on in the classroom and the teachers still be presenting a view of mathematics as a certain and definite body of knowledge of which he or she has possession, and the purpose of the investigation is to discover the right way to do it, or to get the teacher to give the right answer and the correct method. Similarly, the lesson may have a more formal appearance, perhaps even a drill exercise, but in context, where the approach is one whereby pupils are actively involved in the processes of doing mathematics, hypothesising, testing, etc., and the particular lesson is aimed at the acquisition of skills required, and seen by the pupils as needing to be acquired, to be able to continue with their mathematical work. It is however more likely that a teacher using a more 'open' way of teaching would use investigative, problem-solving materials and methods. Similarly, it is also likely that some, at least, of the lessons would be individualised learning, if not the whole approach. This school was chosen in the

knowledge that the overall programme was not one of individualised learning, although it was not known whether any investigative work took place at some stages in the school mathematics. For practical reasons of the ease of observation and reliability of judgements of categories of teacher-pupil interactions, it was decided to observe in a school using more traditional classroom techniques. It seemed possible that within such a system there would be opportunity for variety in ways of teaching of individual teachers. By focusing on those categories of teacher and pupil behaviour which were proposed as revealing an 'open' approach, the intention was to be able to identify any tendency in this direction on the part of the teachers.

The results of the study do not suggest that this assumption was incorrect. It is proposed here that one could not maintain that the uniformity that appears amongst these teachers is as a result of the inapplicable nature of the assumption. One could attribute the lack of teacher questions of categories 3 and 4 to factors other than the school, such as lack of skill on the part of the teachers themselves. The size of the group of teachers observed in this study, 9 in all, would tend to suggest that this is not the case. Taking the study to other schools with different programmes of school mathematics, as well perhaps as using a different observation scheme with more details being picked out, such as SCAN (Beeby 1979) would test this further, and this is thus one of the options for a second stage study.

(b) Interview

The interviews were short, and intended to record the mathematical background, teacher training, and teaching experience, and to give the teachers the opportunity to expand on any comments about the questionnaire, and any contradictions in answers. The interviews were not recorded, and this is a disadvantage. Notes were taken, and in a few instances, verbatim comments were written down and are included in section 7.1. The original intention was to not permit the interviews to become a substantial part of gathering information on teachers' views, and this was maintained, despite it becoming obvious after a few

interviews that a different approach could be taken. Thus, clearly the interviews could have been expanded into a major part of the research, as it has in other research reported here, and this could entail a restructuring of the study towards a sociological study of the influences, relationships, power-structure etc. within the department and the school. This is an important option for further research.

(c) The Questionnaire

Specific comments on the questionnaire will be made first, followed by a discussion of its overall value.

Some of the statements could have been improved, e.g. one could maintain in items 13 and 16 that teachers are experts and hence the statement can have different meanings. The great majority of the statements however did not appear to create any problems of interpretation or lead to confusion. The questions were sorted out by the researcher and not on any statistical basis. This is a possible source of criticism. Hence further work could go on in the direction of refining the questionnaire. Factor analysis is one possibility, although it is a highly subjective method because of the choice of correlates. Another more objective possibility is Item Analysis. In both these, and other techniques, a wider sample of completed questionnaires would be required and this is again an option for further research. As a method of examining teachers' views, or revealing the implications of teachers' views to the teachers themselves, or in initial teacher training in order to deal with this area of teachers' views of mathematics, this questionnaire may have value, and hence further development of it would be important.

The marking scheme, allowing for example 1 for strongly agree, 2 for agree, 2½ for no view, 3 for disagree and 4 for strongly disagree, can be criticised for implying that the gap between strongly agree and agree is the same as between agree and disagree. This is probably not the case. For a mark of 1 to distinguish between strongly agree and agree, and for the same amount to show a change to disagree is difficult to justify. However, if one restricts one's analysis to exclude the

strongly held views, for the reasons discussed in 7.2.1, this problem is avoided. Any further use of the marking scheme with the questionnaire, using the strongly held views as well, would have to take this into account. Similarly, assigning a mark of 2½ to 'no view' is appropriate when not using the strongly held views categories, but more suspect when these are included.

On the question of the 'no view' category, the problems suggested in 6.2.1 stand here, in that including this option has disadvantages, as does not including it. Forcing teachers to make a choice by not allowing this option results either in a false set of answers, or at least a number of statements which are suspect, or, as was found in the earliest pre-tests, teachers make that choice for themselves by choosing not to answer. Including the option provides a possible easy way out for teachers who do not want to answer a particular item, which could happen for many reasons, e.g. unwilling to commit themselves, unwilling to think the statement through because of tiredness, etc. It was decided on balance to include this option, but the reservations recorded here, stand.

The general value of this research tool will best be discussed from two possible criticisms, that of the value of a questionnaire for this stated aim, and the relevance of assigning a mark to a teacher's view. The problem, firstly, is that to get at the views of teachers on aspects of their work that are more theoretical and philosophical, without alienating the teachers from the task, making them feel that their knowledge is inadequate to answer the question, or allowing the response that the teacher does not have a view of the nature of mathematics, is very difficult. If teachers generally hold views that are scarcely coherent, direct questions about their philosophical opinions are quite likely to lead to one or other of these negative responses. A major influence in this thesis is the desire to examine and analyse the options open to teachers as to the nature of mathematics, and to make explicit the implications of holding those alternative views. As a consequence, general, vague questions about teachers' attitudes that might be used in order not to alienate those teachers are not

appropriate in this context. If the alternatives are delineated, the questionnaire would need to be constructed to reflect this, or the questions asked in an extensive interview, with all the dangers outlined above that this might imply. Consequently, with the reservations put forward here, and others that may be proposed, it still seems appropriate to use a questionnaire of this type, for this purpose.

Finally there must be some doubt as to the significance of assigning a mark as an overall representation of a teacher's view of mathematics and mathematics education. This criticism can be accepted without completely rejecting the idea of evaluating a teacher's answers. Within the context of the continuum outlined in 4.2, marks can provide a ranking of a number of teachers that might be significant. A mark could also provide an indication of the position of an individual teacher, without too much value being placed on the exact numerical value.

7.2.3 Discussion and Rationale of Second-Stage of the Study

Three main items emerge from the first stage of the field study, as potential areas of a second-stage:

- (a) further examination and improvement of the questionnaire, since the questionnaire could have applications and uses in a variety of areas: in revealing teachers' views of the nature of mathematics and mathematics education; perhaps as an introduction to these issues in initial training and in-service training of mathematics teachers; research on teachers' views other than the present one (two research students so far have used this questionnaire in their own work, in 1985/86 academic year).
- (b) further work based on the hypothesis that the depth of teacher questions (categories 3 and 4), of pupil responses (categories 8 and 9) and teacher reactions to those responses, are significant indicators of the different ways of teaching mathematics discussed in this thesis.
- (c) further research to examine the hypothesis that other factors, which may be called the specific school context, are stronger determinants

of mathematics teacher behaviour than teachers' views of the nature of mathematics. For example, a further stage of study could be to probe the teachers in this school, perhaps by interviews, to reveal some of these influences.

For the second-stage of the study, it was decided to focus on (a), further work on the questionnaire. It has been proposed that the questionnaire is an instrument that could be used in at least three main areas: other research on teachers' attitudes to mathematics; use with students in pre-service training; and use with teachers in in-service courses. Hence, with these applications in mind, and in particular its use with students, it was decided to further examine the questionnaire. For this reason also, the audience that was chosen for this, was a large group of Postgraduate Certificate in Education students, of Secondary Mathematics.

It was also decided to interview a small sample of the students who, after analysis of the questionnaire, were seen to be at the two extremes of the range. This exercise was designed to examine the possible reflection of views of mathematics in the practical field of the conduct of lessons. It was clearly even more important to avoid the 'specific school context' in this exercise with students, since their inexperience in teaching combined with their concern with being assessed by the teachers of the school and their tutors would be likely to result in teaching behaviour that was not of their own free choosing. Thus it was decided to provide some stimulus material such as a video-tape or audio-tape extract of a mathematics lesson, and interview the students to identify their impressions of the lesson. If there were clear differences in the types of criticisms and comments on the lesson, that corresponded to the results that the interviewees obtained on the questionnaire, this would be quite strong information in support of the value of the questionnaire, and slightly weaker support for one of the hypotheses of the thesis, that there is a correspondence between views of mathematics and ways of teaching.

The full description of the second-stage study follows in Chapter 8.

CHAPTER 8 - THE SECOND-STAGE STUDY

8.1 The Programme of the Study

The intention of this second stage was, as described in 7.2.3, to examine further the questionnaire, for its usefulness in distinguishing the views of teachers, and the possible reflection of those views in the framework of the continuum of 'open' - 'closed' ways of teaching. The study formed two parts:

(a) First, the questionnaire was given to a larger sample. It was decided, for this purpose, to use a group of Postgraduate Certificate of Education in Secondary Mathematics students at the beginning of the second term of their course. The reasons for this decision were: this was a large group, in one place, and hence distribution, completion and collection of the questionnaire would be convenient; the follow-up work would be facilitated considerably by virtue of the group being together, in a place where video facilities would be available, and suitable times not difficult to arrange; the students would be likely to have a positive attitude to the study, and, finally, this group would be free of the influence of the 'specific school context' described in the preceding chapter. In addition, it emerged after discussion with the tutors of this group that the tutors would wish to use the fact that all the students had completed the questionnaire to initiate some work on attitudes to the nature of mathematics, after the completion of the study. This in itself would be a use of the questionnaire, as suggested in 7.2.3. A possible disadvantage of using this group would be that they might be conscious of what they imagine they 'ought' to say, as students, and as the particular students of their particular tutors. The questionnaire was completed by the students at the beginning of the second term, in January 1986, after the Christmas holiday and after a first term in which the latter part had been spent on teaching practice. Thus it was hoped that these influences were still fairly weak. The first stage of the study also revealed that the teachers were aware of the 'expected' answers, and conscious of the university-based

researcher, and hence these influences will always be present, to some degree. The results were then be subjected to Item Analysis, a statistical technique which is described fully below, in 8.3, to identify the items in the questionnaire that do distinguish teachers' views. within the assumptions at the foundations of the construction of the questionnaire.

(b) Second, a small sample of the students was shown a video extract of a mathematics lesson, and asked to comment on it. The students were chosen in two groups, one from amongst those scoring highest on the questionnaire, and the other from amongst those scoring lowest. It was decided to choose an extract that showed an experienced, able teacher, teaching in a stimulating but relatively 'closed' way. It would then be interesting to note any differences in reactions to the lessons, and in particular any correspondence. This is described in 8.4 below.

8.2 The Questionnaire

At a meeting with the tutors of the group of students, arrangements for the administration of the questionnaire were outlined. The last half-an-hour of a seminar was to be used, each tutor-group would complete the questionnaire in the group, without any discussion, and then return them all to the tutor, who would pass them to the researcher. As introduction to the work, the following was given to each tutor, to read out to the group:

"This questionnaire is being developed, as part of Steve's doctoral thesis, to examine teachers' and student teachers' views of mathematics and mathematics education. It is not meant to be related to your teaching practice experience at all, but to your own views of mathematics teaching.

Please fill in your name on the cover. Most of the items simply require a tick in an appropriate column. There is space at the end of the questionnaire for any comments you may wish to make after completing it.

The questionnaires will be treated with complete confidence and in any reporting, no names will be mentioned.

Thank you for your cooperation and help in this research."

In all, 42 students completed the questionnaire, out of a group of 48, these 6 others being absent on that occasion. The sample consisted of 24 women and 18 men, of whom 33 were mathematics graduates, the others having at least one year of degree level mathematics in their qualifications. 17 of the sample were aged over 22, and had thus had other experience beyond university, including having children, employment in industry or business, etc. The results of the questionnaire are given in Table 8.2.

It was decided to use exactly the same marking scheme as was used in the first-stage study, which includes no marks for items 1,2,8,16 and 20. These were originally included in the questionnaire for completeness, but were not assigned marks as they were considered not sufficiently discriminating, from the informal and pilot testing. For the Item Analysis, since the role of the particular statistical technique is to examine just this, these items were given marks and included in the analysis, to corroborate or contradict that view of those items.

Many more 'strongly agree' and 'strongly disagree' responses were given, and hence the range of possible marks from 30 to 120 is more appropriate here than it was with the teachers in the first- stage study. It will be seen that the marks were from 68% to 100, which, it may be suggested, is quite a wide spread, as compared with the spread of marks of the teachers in the first-stage study, which was from 65 to 86%, although this may be simply due to a larger sample. The extension of the top mark from 86% to 100, whilst the bottom mark is very close, together with the mean and median marks being 85, almost the maximum mark of the first group, suggests that the students who completed the questionnaire in this second-stage are generally closer in views to the 'fallibilist' or 'open' end of the continuum.

Table 8.2 Results of the Questionnaire

No. in ranked order	Mark for 30 questions		
1	100		
2	99		
3	95½		
4	95		
5	94½		
6	93½		
7	93		
8	93		
9	93		
10	92½		
11	92		
12	91		
13	89		
14	89		
15	89		
16	88		
17	87		
18	86½		
19	86½		
20	85½		
21	85½		
22	85		
23	84½		
24	84½		
25	83		
26	82½		
27	82½		
28	82½		
29	82½		
30	81½		
31	80½		
32	80		
33	79½		
34	78½		
35	77		
36	75½		
37	75		
38	75		
39	73½		
40	70½		
41	70		
42	68½		
Total	3570	Mean 85	Median 85.25

Many students wrote comments at the end of the questionnaire. These comments have, where appropriate, been grouped together below. They consisted of some general comments, as in (ii) on whether the questionnaire was descriptive or prescriptive of schools today, and in (iii) on the difficulty of wording, and some specific comments on particular items, as in (i) and (iv). Finally, the remainder are grouped in (v).

(i) Six students wrote comments on item 16 "Public examination syllabus, style etc. should be decided by experts.", to the effect that 'experts' could be interpreted in various ways.

e.g. "I agree that syllabuses should be decided by experts - but experts of the children and what they are going to use in life, not experts of mathematics (university professors etc!)"

"Experts = teachers/pupils etc. or do you mean experts in the sense of highly successful mathematicians - My answer makes the first assumption"

(ii) Several were confused as to whether the questionnaire was to be completed in the light of how schools are today, or what the person thought schools should be.

"When I started the questionnaire I was in doubt about whether I was to give what I thought actually happens in school or what I thought ought to happen in school - especially question no. 2 and 8, and 4, 11, 12."

"Many questions are ambiguous - Does it require what we think the case should be or how we act within current constraints."

It had been the intention of the phrase "...your own views of mathematics teaching" in the instructions to be read out by each tutor, that there would not be this confusion. On reflection, it could have been made more clear.

(iii) Some students found some of the wording difficult.

"I, as a mathematician feel that the questionnaire was written in a difficult language and was not as concise as it could have been. Was it possible to have produced the statements in a more simple manner."

"You need a degree in English to understand some of these questions, something I do not possess."

"Found some of the wording a bit misleading and some questions tended to lead towards a 'correct'ish response. Generally there were words which I didn't know the true meaning of!"

(iv) One student questioned some of the section 2 items.

"Question 29 - Not a theorem until proved.

Question 32 - Different truths between different mathematical structures (e.g. Euclidean and non-Euclidean geometry). However within the formal structure provable statements are inevitable."

v Various other individual comments such as:

"Questionnaire very thought provoking - makes one stick to conclusive ideas about aspects of mathematics and mathematics education without being able to "straddle the fence". Questions themselves are very pertinent and don't sidestep the issue."

"You obviously have a high regard for us, your subjects, and expect us to have deep philosophical thoughts. None the less it is interesting and makes me think a lot more than I normally (sic)!"

"Would have liked to expand on some answers - Quite frustrating!"

These comments, taken together with the results of the Item Analysis, contribute to the review of the wording of the questionnaire which follows.

8.3 Item Analysis

This procedure, described in, for example, Allen and Yen (1979), is a screening technique. It is an approximate method for examining the discriminatory facility of items in a questionnaire. It involves taking

the mean mark of groups of the sample, in their ranked order, for each item, and plotting these means. Where the graph shows a decreasing function, the item is a good discriminator, where the graph is flat, it is not, and where it is erratic, some error in construction of the item, due perhaps to poor wording for example, has been the likely cause. Seven groups, each containing six people, were used, in order to strike a balance between too few groups and too large ones, with the constraint of the factors of the size of the sample, 42. The table of means, and the graphs for all 35 items, form table 8.3(a) and figure 8.3(b).

As mentioned above, marks were given for the items which were not included in the results of the questionnaire, either in the first-stage study with the teachers in the school, or with this larger sample of PGCE students. This is to enable a proper examination of the discriminatory facility of those items also. The marks given for those items are shown below:

	Strongly Agree	Agree	No View	Disagree	Strongly Disagree
1)	1	2	2½	3	4
2)	4	3	2½	2	1
8)	1	2	2½	3	4
16)	1	2	2½	3	4
20)	1	2	2½	3	4

Thus the Item Analysis was applied to all 35 items in the questionnaire.

The items which are shown as being good discriminators, for this sample, may be seen from table 8.3(a) and figure 8.3(b) to be as follows:

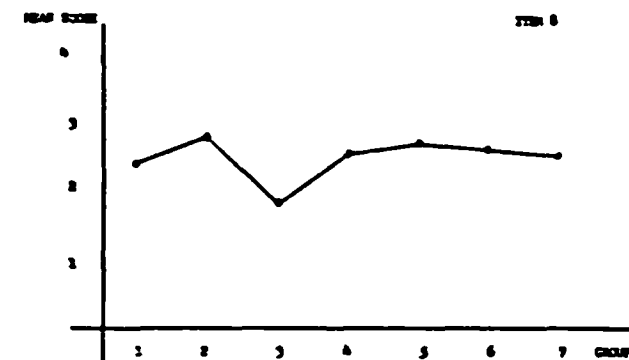
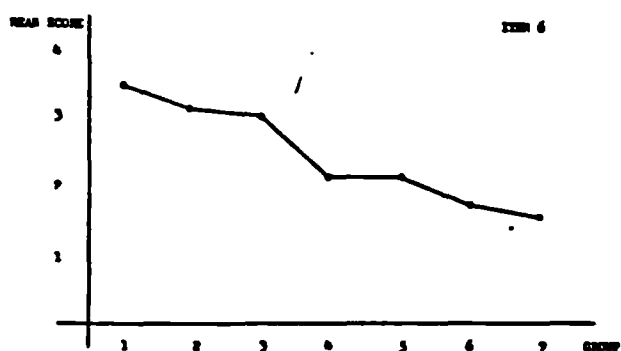
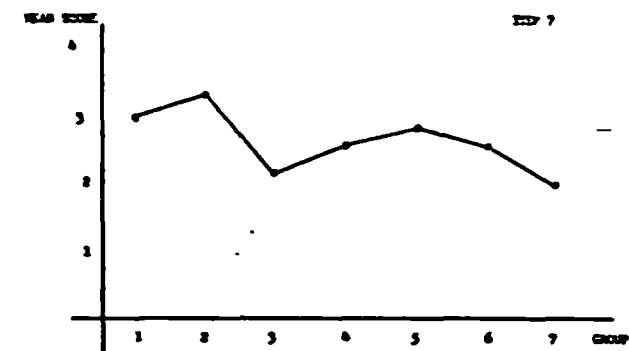
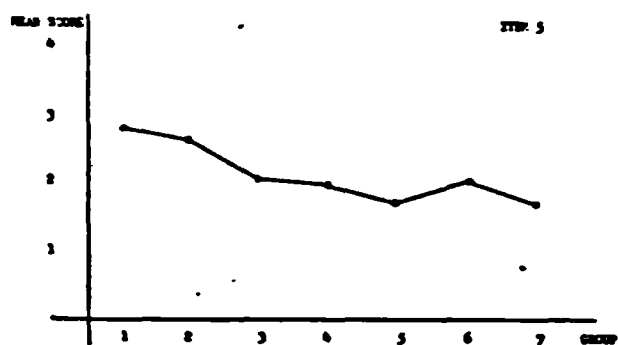
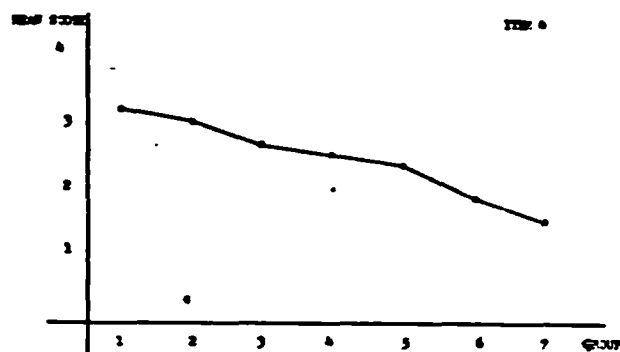
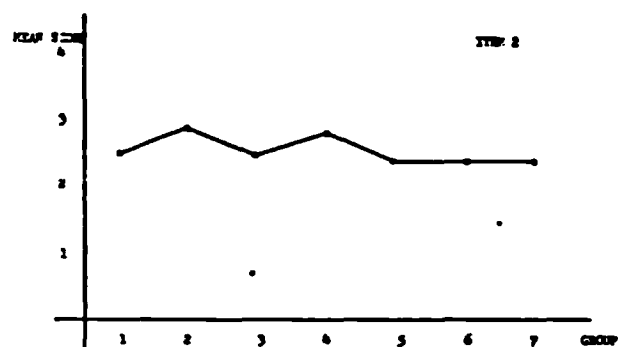
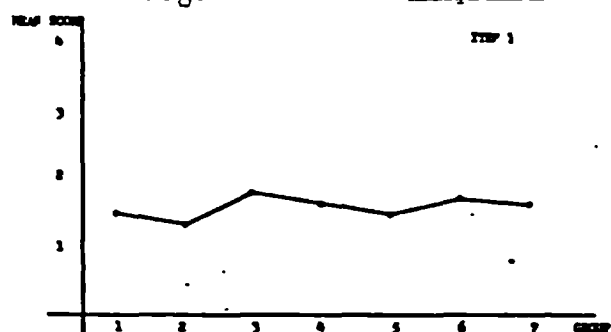
3, 4, 5, 6, 9, 10 (weak), 11, 12, 14, 17, 18, 19, 21 (weak), 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34.

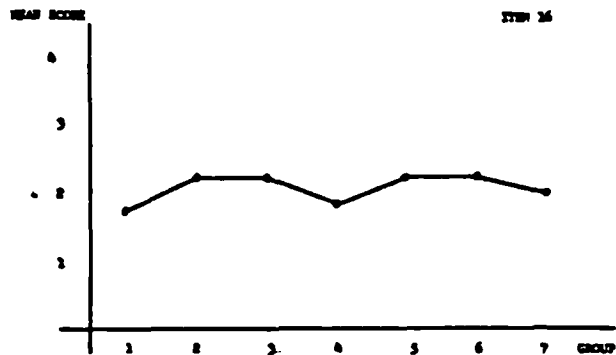
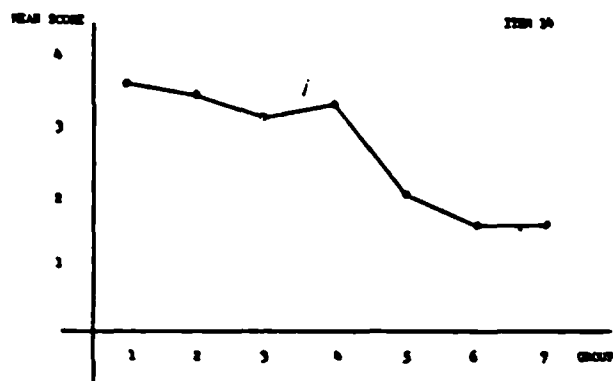
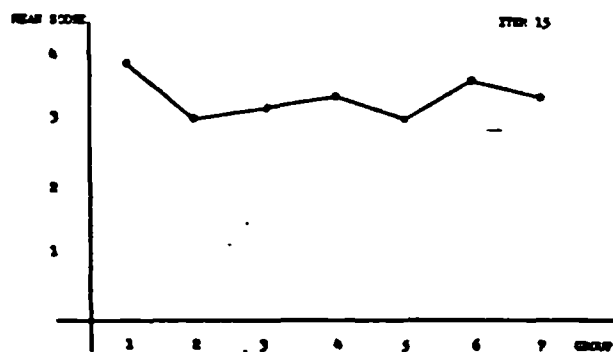
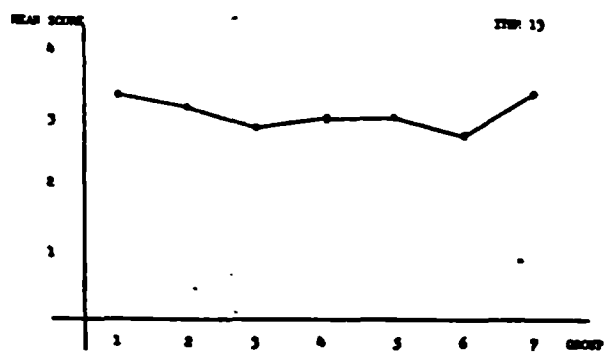
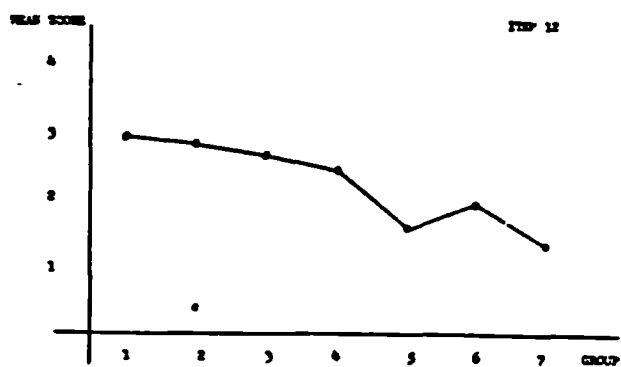
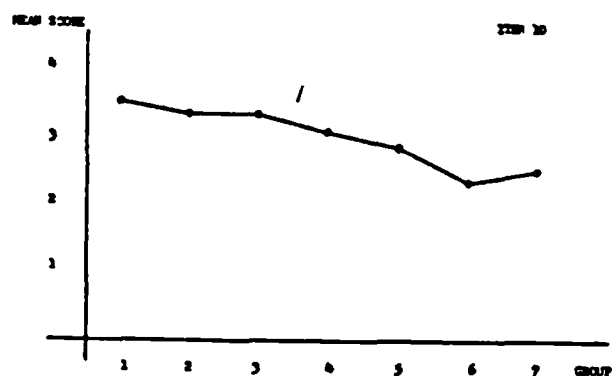
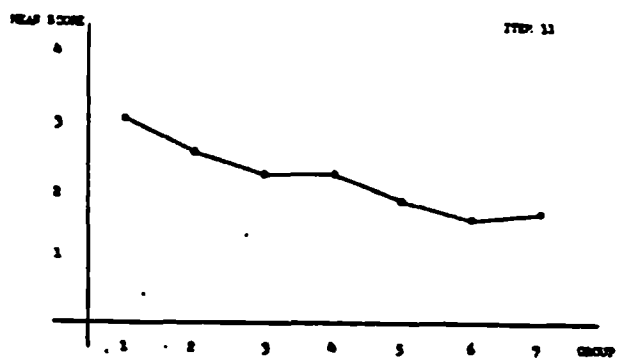
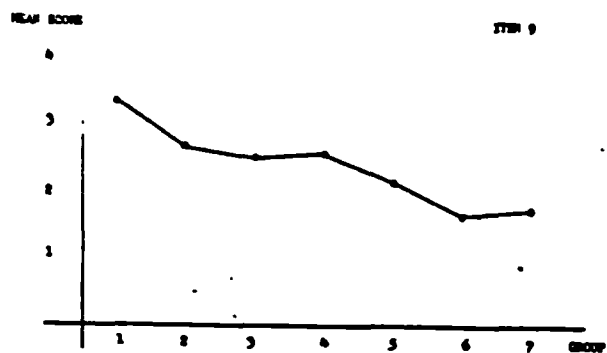
The items denoted 'weak' are decreasing functions, but with a very small decrease overall.

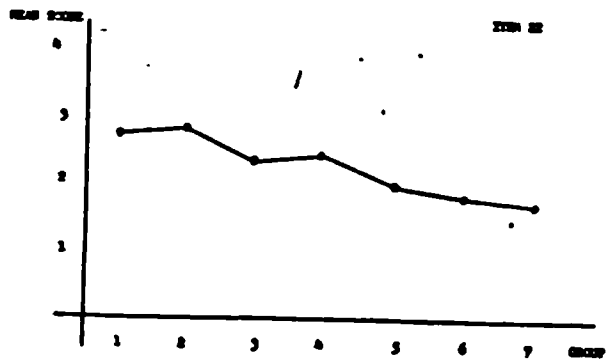
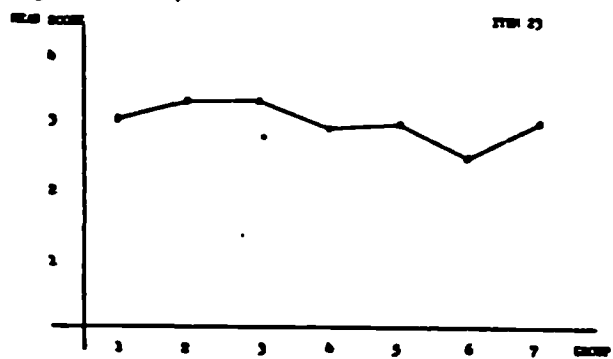
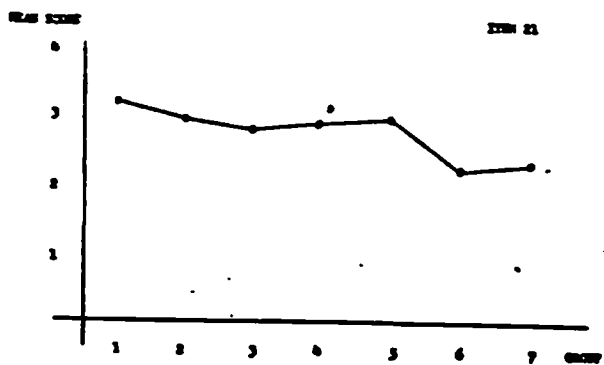
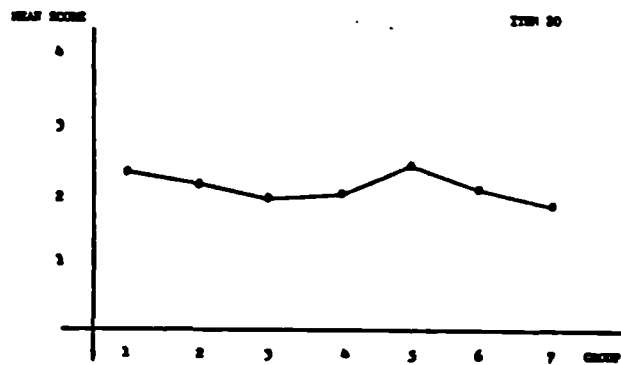
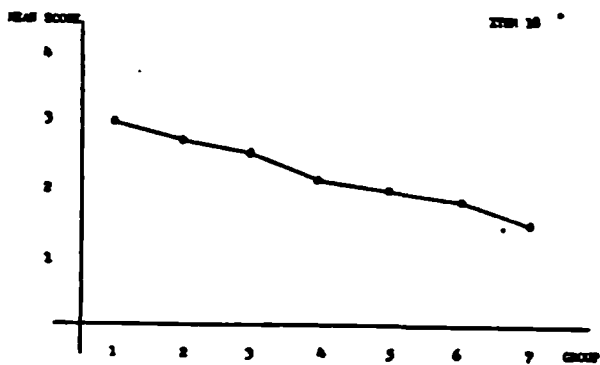
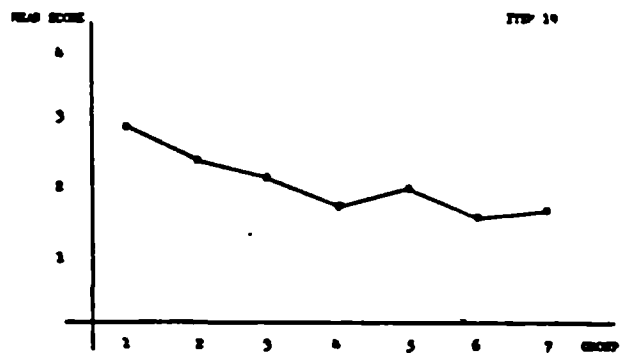
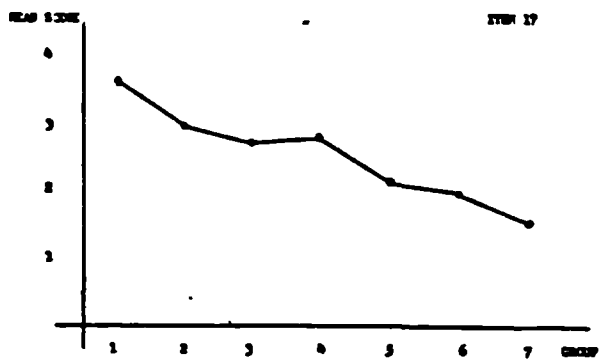
Table 8.3(a) Mean Mark for Groups of Students in Order of Ranking

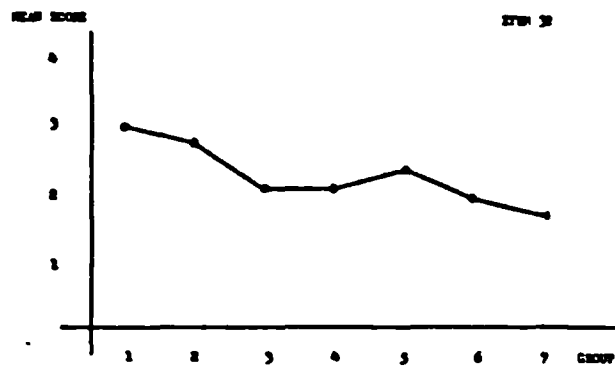
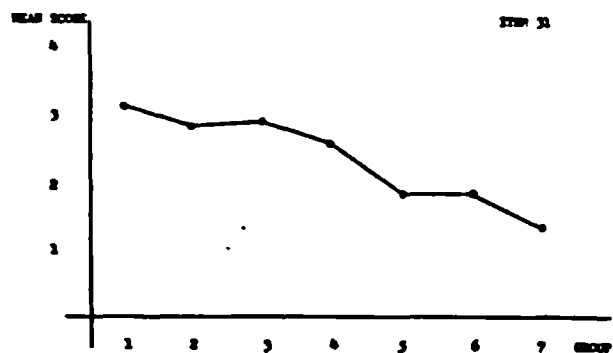
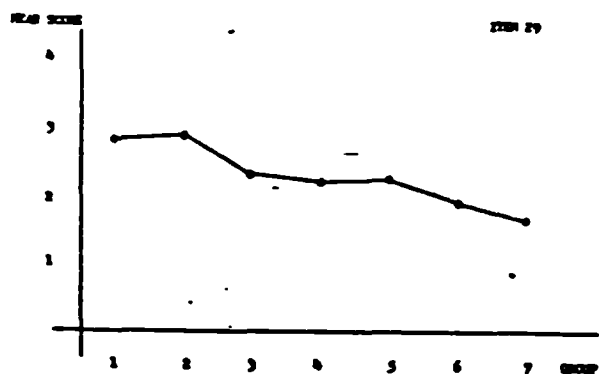
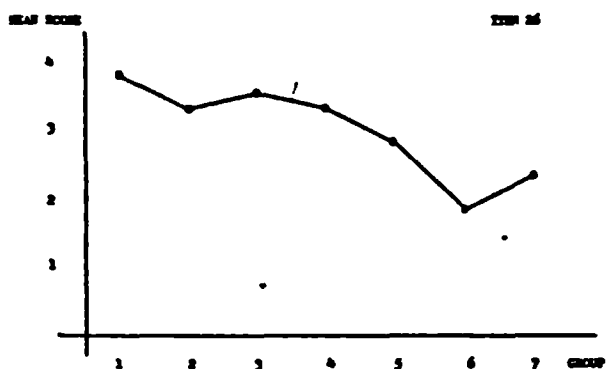
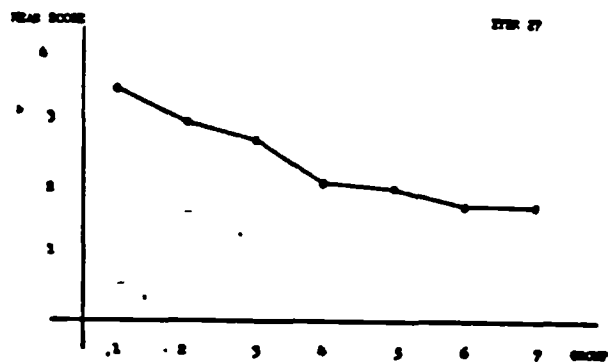
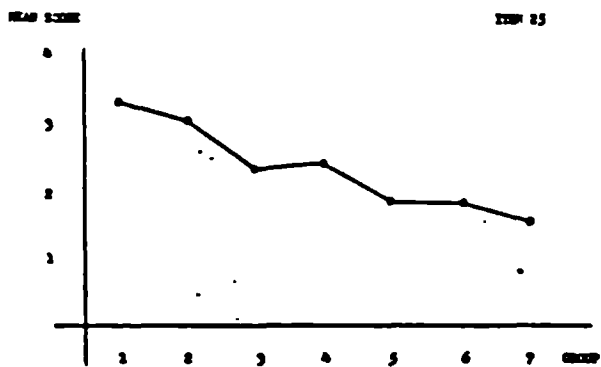
Question no.	Gp. 1	Gp. 2	Gp. 3	Gp. 4	Gp. 5	Gp. 6	Gp. 7
1	1.5	1.33	1.83	1.67	1.5	1.75	1.67
2	2.42	2.83	2.42	2.75	2.33	2.33	2.33
3	2.92	2.17	2.33	2.17	2	1.92	1.5
4	3.17	3	2.67	2	2.33	1.83	1.5
5	2.83	2.67	2.08	2	1.75	2.08	1.75
6	3.5	3.17	3.08	2.17	2.17	1.75	1.58
7	3	3.33	2.17	2.58	2.83	2.58	2
8	2.42	2.83	1.83	2.58	2.75	2.67	2.58
9	3.33	2.67	2.5	2.58	2.17	1.67	1.75
10	3.5	3.33	3.33	3.08	2.83	2.33	2.5
11	3.08	2.58	2.25	2.25	1.83	1.58	1.67
12	2.92	2.83	2.67	2.42	1.58	1.92	1.33
13	3.33	3.17	2.83	3	3	2.75	3.33
14	3.67	3.5	3.17	3.33	2	1.58	1.58
15	3.83	3	3.17	3.33	3	3.58	3.33
16	1.75	2.25	2.25	1.83	2.25	2.25	2
17	3.67	3	2.75	2.83	2.17	2	1.58
18	3	2.75	2.58	2.17	2	1.83	1.5
19	2.92	2.42	2.17	1.75	2	1.58	1.67
20	2.33	2.17	1.92	2	2.42	2.08	1.83
21	3.25	3	2.83	2.92	3	2.25	2.33
22	2.75	2.83	2.33	2.42	2	1.83	1.75
23	3.08	3.33	3.33	2.92	3	2.5	3
24	3.58	2.92	1.92	1.92	2	1.67	1.67
25	3.33	3.08	2.33	2.42	1.83	1.83	1.58
26	3.83	3.33	3.58	3.33	2.83	1.83	2.33
27	3.5	3	2.75	2.08	2	1.75	1.75
28	2.5	2	2.25	1.75	2.25	1.5	1
29	2.83	2.92	2.33	2.25	2.33	1.92	1.67
30	2.83	2.67	2.17	2.17	2.17	1.75	1.67
31	3.17	2.83	2.92	2.58	1.83	1.83	1.33
32	3	2.75	2.08	2.08	2.33	1.92	1.67
33	3.25	3	3.08	2.75	2.17	1.67	1.17
34	2.83	2.75	2.33	2.42	2	2.08	1.25
35	2.58	2.67	4	2	2.58	4	2
TOTALS	105.4	98.08	90.23	84.5	79.23	72.39	65.15
Overall Total	594.98 * 6 = 3569.88						
Rounded off 3570							

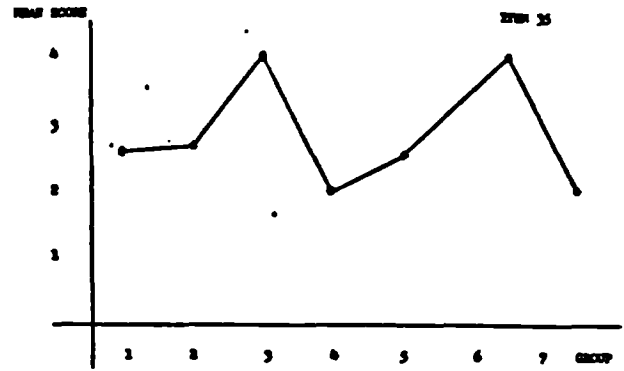
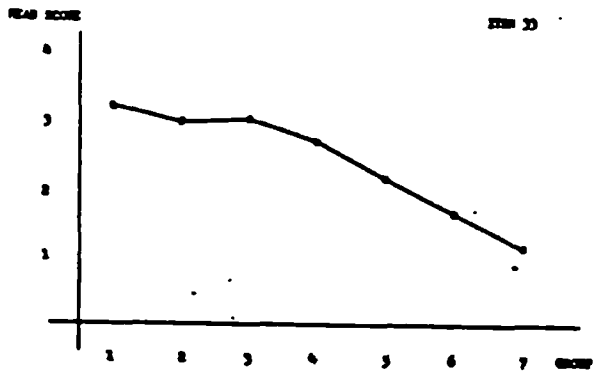
Figure 8.3(b) Graphs of Item Analysis for Items in Questionnaire











Each item which was not a good discriminator is listed below, and tentative reasons for this inadequacy are given in each case.

- 1) "The examination syllabus largely determines the school syllabus."

This is one of the items originally excluded from the marks. It probably falls into the category mentioned by a number of students as depending for a response on whether one thinks in terms of 'does' or 'should', and since interpretations will differ, this item is not useful in this form. It could be changed to become 'should largely determine' in the wording, and the item might then be more discriminating.

- 2) "Learning mathematics in school is essentially an active process."

This item is probably misunderstood for one of two reasons, again on the should/is alternative, and also on the possible confusion in the meaning of 'active process'. I intended this to be taken, together with item 8, a 'passive process', to be a choice between either a Piagetian view or an empiricist or platonic view of the learning process. It may be that if this is explained more in the statement, such as for example "Children learn mathematics by a passive process of the absorption of knowledge", it could be used.

- 7) "The examination syllabus should not be the main factor in determining the school syllabus".

This item clearly states 'should', and so fails for a different reason to the allied item 1. There are two parts to the graph of this item which show increases, whilst overall there is a decrease. Perhaps this has suffered from an aberration of the particular sample. One might have expected a general agreement across the population. Its reason for failure to discriminate is not clear.

- 8) "Learning mathematics in school is mainly a passive process."

This has been discussed with item 2 above.

- 13) "Teachers should be able to influence public examination boards in the style of examination, syllabus content etc."

The graph of this item shows simply that almost everyone agrees, and thus it is not a good discriminator.

- 15) "The development of problem solving skills in pupils should be seen as an essential part of school mathematics."

Again, this item gained general agreement.

- 16) "Public examination syllabus, style etc. should be decided by experts."

As discussed above, a number of students specifically mentioned this item as ambiguous, focusing on the word 'expert'. This item could possibly be improved by writing 'mathematics experts'.. Clearly, though, all the items referring to decisions about the school mathematics syllabus, the examinations, who decides, and which influences which, need rephrasing.

- 20) "Mathematics is training you to be logical."

This is the fifth item that was originally excluded from the marking scheme. There is no doubt that learning mathematics trains one to be logical in a mathematical sense, but there are other senses to the term logical, and this may be the cause of the failure of this item. The sense of the intention of this item may better be conveyed by "Mathematics is the best subject to train you to have a logical mind."

23) "Asking open questions is vitally important as it gives pupils the opportunity for creative thought."

This item has general agreement amongst the sample. This is true also of the teachers in the first-stage study. This may be a sign of familiarity with the 'correct' responses.

35) "Mathematics is essentially hierarchical and cumulative. Although progress does go on making earlier work more rigorous, generally new knowledge builds on former work."

Answers to this item are erratic. It is not simply a poor discriminator. The intention of this statement was to offer a cumulative view of mathematical knowledge, whose alternative is a Kuhnian view. In a more straightforward sense of absolute truths or fallibility, this idea is covered in the more successful items 29 to 34, and thus could be omitted.

In conclusion to this discussion of the questionnaire, most of the items are sufficiently discriminating to be valuable, and some of the others could perhaps be adapted to be included. All the items which had originally been excluded from the marking of the questionnaire showed themselves to be poor discriminators. Since the 'strongly agree' and 'strongly disagree' marks have been included as significant in this sample, as these were used often by the students in their responses, the early criticisms of the validity of the marking scheme apply. However, as discussed earlier, the value of the questionnaire is not in the assignment of a specific mark as an indication of the views of an individual.

With this criterion established, it can be proposed that the questionnaire's overall value has been corroborated by this statistical exercise.

8.4 The Programme for the Interviews

After examination of many video-recordings of mathematics lessons, one was chosen which was deemed suitable, from the following points of view:

- (a) the extract lasted 5 minutes, which seemed an adequate amount of time for impressions to be formed, without providing too much information,
- (b) the extract was 'complete' in the sense that it covered the introduction to some work, the development, and an ending, when the pupils worked alone,
- (c) one's initial reaction to the classroom, the pupils and the teacher was one of an attractive situation,
- (d) the teacher's choice of introduction was a novel and interesting one, which tended to mask a quite didactic way of teaching.

The extract chosen was of the introduction to the solution of simultaneous equations to an average ability third year set. The teacher put up on the blackboard:

c = cups of coffee

t = cups of tea

$$7c + 2t = 38.$$

She then proceeded to invite the class to offer possible solutions to the prices of a cup of tea and a cup of coffee, given that the equation meant that seven cups of coffee and two cups of tea could be bought for 38p.

It is to be noted that no students noticed the major error made by the teacher here, in her definitions of the variables c and t. It is not surprising, given incidents like this, that pupils have difficulties with the concept of variable. However, this was not the focus of the interview.

After receiving several suggestions, including one or two incorrect ones, she then put up another equation, saying that someone else had bought some cups of tea and coffee. This was

$$2c + 3t = 23.$$

She invited the class to say which answer given for the first equation fits the second. One student offered the correct answer, and the lesson ended when the teacher asked how the pupils could satisfy themselves that this was the only one that fitted, and she asked them to try some of the other solutions to the first equation.

The example the teacher used is an interesting and 'real' one, in the sense that it is probably meaningful to the pupils to buy cups of tea and coffee and pay an overall price, although the answers are extremely unreal for the prices of tea and coffee in 1986. The lesson was a very recent one. On the other hand, bringing some cups to the table, giving the total price, and then asking for possible answers for prices is not a usual situation. However, the friendly way the lesson is conducted, and the interesting choice of model, even with its mathematical flaw, for the simultaneous equations masks the quite didactic teaching method used, and this was the main reason this extract was chosen for the interviews.

An interview protocol was developed, and tested on a colleague. The full protocol is given in figure 8.4. After an assurance of confidentiality, the programme of the interview, namely to show an extract of a mathematics lesson and ask the interviewee for comments, was clearly stated. Then, to allow that idea to be absorbed, and to help the interviewee relax, a question was asked regarding the interviewee's teaching practice period. After a short exchange, the interviewee was drawn back by a repeat of the programme for the interview, and the video was then shown. Immediately afterwards, a cassette recorder was switched on and the interview took place. Questions were asked only when it was felt that the interviewee needed prompting, and as can be seen from the complete transcripts, which form Appendix D, phrases used

by the interviewee were generally repeated. Occasionally it was felt that the interviewee needed to be focused on the way in which the teacher had chosen to deal with the topic, and a suitable question was asked. When the interview finished of its own accord, the tape was switched off, and a final question asked, to bring the interviewee out of the session.

Figure 8.4 Interview Protocol

Thanks for helping me with my research. Let me just say first of all that this whole interview will be treated in complete confidence. I would like to record our conversation on tape, but I'll erase it afterwards if you wish.

In a few moments I'm going to show you a video of an extract of a maths lesson, and then I'm going to ask for your comments and views on it.

Just before we start, though, tell me how does it feel now that you have finished your teaching practice?

(short conversation)

O.K. Let's have a look at this video of part of a mathematics lesson, and afterwards I shall ask you what you thought.

(video)

(switch on cassette recorder)

Right, what did you think of that lesson?

(interview)

O.K. Thank you.

(switch off cassette)

What's it like to observe a lesson now, after all your teaching practice experience?

8.5 Results of the Interviews

Four students were interviewed, A and B had scored the lowest marks in the questionnaire, and C and D had the two highest marks, of the whole group. It can be seen that A felt that the teaching method was too open-ended, and that the teacher would have been more successful if she had directed the class more. She said "...I think it would have been better if she had given them more hints, more guidance. It was much too open-ended."

Interviewee B felt that the amount of direction given by the teacher was about right, but that she could have asked more direct questions of the pupils, in particular the pupils who were perhaps not paying attention, or had not understood. He said "She could have prompted others who were just quiet, or she could probably see that some of them weren't even following the class, following the lesson. So I mean she could have prompted them to attention. Sort of re-focused their minds on the lesson as opposed to their own things. Yeah, I think that was the main thing... From the beginning I thought there was too much silence, too little interaction."

C was not too clear at first as to what she didn't like about the lesson, but it emerged that she felt that the teacher didn't do enough with the idea. She said "When she went on to the second I said well alright she's going to link it up now, and all she did was to say if I wrote that and that equals that, which one of these will fit in? You know what she could have done, she could have said - she could have done the second equation exactly like the first, and let them throw out about 10 different solutions, and then say, right, now what do you notice? And if that goes, then say this is what we mean when we say solving equations simultaneously."

D liked the idea for introducing simultaneous equations, and also liked the way that the teacher encouraged all answers even the ones that turned out to be wrong. She picked on the idea that the teacher was

very directed. "I don't think she was too open. In some ways she was very directed. She knew what she wanted but on the other hand she did accept - she did encourage all the possible solutions" She then went on to suggest the same approach as C suggested with the second equation.

Both interviewees C and D seemed to have been concerned with pupils getting to grips with the meaning of solving equations simultaneously, each equation having at least a large number of solutions on its own, but there being just one solution that fitted both. Their appreciation of some aspects of the teacher's approach, and way of teaching, did not preclude their criticism of the teacher, in terms of their view of the aims of teaching mathematics, that is, to introduce the students to the concepts involved. The other interviewees, A and B appeared to have been both mostly concerned with whether the teaching method was detailed enough to elicit the right answers from the pupils. They too were obviously impressed with some aspects of the lesson, in the main the choice of model for simultaneous equations, and the teacher's openness to wrong answers, but were also critical of other aspects. They were both, however, very concerned that the teacher had not been explicit enough about the procedure that the pupils must learn, in order to find different solutions to the first equation.

8.6 Conclusions of Second-Stage Study

The Item Analysis has enabled the items in the questionnaire to be examined in detail for their discriminatory facility, and a full discussion of appropriate changes to the items was carried out.

The interpretation of the interviews described here appears to lend weight to the hypothesis that the questionnaire distinguishes and identifies a fallibilist view of mathematical knowledge, together with an 'open' way of teaching mathematics.

CHAPTER 9 - SUMMARY, REVIEW AND LIMITATIONS
AND
IMPLICATIONS FOR FURTHER RESEARCH

9.1 Summary and Review

This thesis consists of three sections, an examination of the situation in the philosophy of mathematics, some proposals regarding the connections with the teaching of mathematics, and field study in two parts, to examine some of the consequences of the thesis in the practice of mathematical education.

In the first section, it was proposed that a relativist view of knowledge, in the sense of the 'strong programme' in the sociology of knowledge, far from being restrictive, or even destructive, to the progress of thought, is a challenging and dynamic one. From an absolutist view, truths are to be discovered, although there are conflicting and competing ways of establishing that a belief or theory is true. There are of course revolutions in absolutist thought, but these are seen as the replacement of erroneous theories by at least closer approximations to the truth, with the model of a critical experiment which distinguishes the two. It can be suggested that an absolutist view has a stultifying effect because of the implicit self-justification of progress towards truth. On the other hand, a relativist approach which sees all knowledge as a social construction, and hence fallible, can be seen to place power in the hands of people, in the sense of determining what is progress, rather than within knowledge *qua* knowledge. An examination of the situation in the philosophy of mathematics followed, which suggested that Lakatos maintained this position in relation to mathematics, if not science, and that this may be seen as a Wittgensteinian view also. It was also proposed that different epistemologies may be seen to provide different forms of social control, viz. the teacher as possessor of knowledge, or as participant with pupils in the language game called mathematics.

In the second section, a review of the literature revealed the need for an examination of competing perspectives of the nature of mathematics, and how they relate to the teaching of mathematics. It was then hypothesised that the two views of mathematics, absolutist and fallibilist, can be seen to be reflected in the teaching of mathematics. One can portray ways of teaching mathematics as a continuum, with the two extremes being an 'open' teaching approach, where children's learning is seen as the creative process of conceptual growth, with the teacher as facilitator in this, mirroring the process of the growth of knowledge, or a 'closed' approach, whereby the teacher sees him/herself as the possessor of absolute knowledge which the children are to acquire, through some adequate form of explanation. Finally, it is suggested that some recent developments in research in mathematics learning theories, namely in the constructivist view, can be seen to reinforce and support the hypotheses of this thesis, that there are two competing views of the nature of mathematics, and that these are connected with and reflected in alternative ways of teaching mathematics, in the sense that an epistemological perspective brings with it consequences for the practice of mathematics education.

In the third section, a study was carried out that attempted to examine some of the consequences of the thesis for the teaching of mathematics. In the first stage, a questionnaire was developed to identify teachers' views of mathematics and mathematics education, containing 35 items, and marked on a Likert scale, and an observation tool adopted, a Flanders instrument. These were then used with the teachers of a particular comprehensive school, with the intention of examining teachers' views and observing their teaching behaviour from the 'open'-'closed' perspective, focusing on the kinds of questions teachers asked, the kinds of responses and ideas that pupils put forward, and the ways that the teachers reacted to these. 9 teachers participated, and a total of 16 classes were observed. It was found that there was a considerable uniformity of ways of teaching, although quite a variation of teachers' views. Hence it would seem that there are other factors that may be more significant in determining the way that mathematics teachers behave. On the basis of the interview data, one could hypothesise that

school-context factors are more likely to influence ways of teaching than attitudes to the nature of mathematics.

Three alternatives were indicated as possible areas of a second stage and of the three it was decided to further examine the questionnaire, and to attempt to observe teachers' ways of teaching, whilst avoiding the influence of the specific school context. There were two stages to this. In the first part the questionnaire was given to a larger sample, 42 students of a Postgraduate Certificate of Education course, and the results examined by Item Analysis. This enabled a discussion of each item in the questionnaire, for its ability to discriminate. In the second part, a small sample of the students was interviewed, after each had seen an extract of a mathematics lesson on a video-tape. The students were asked for their view of the way of teaching they observed. By this method it was hoped that one could avoid the influence of the specific school context, and still provide some information that was concerned with the reflection of views with ways of teaching mathematics. The two students who scored highest on the questionnaire criticised the teacher for being too 'closed' in her teaching, whilst the two students who scored lowest on the questionnaire considered that the teacher was too 'open'. One can hypothesise from these interviews, that there is a strong correspondence between the views of those interviewed, as identified by the questionnaire, and the kinds of comments and criticisms they made of the mathematics lesson extract they observed, in the context of the hypotheses of the thesis and the study.

9.2 Limitations of the Research

The investigation of the teachers' views and ways of teaching in a particular school in the first-stage of the field study necessarily meant making decisions about which school to use, that had implications for the research. Reasons have been given for the choice made, of a school that did not have an individualized programme for mathematics, that was a fairly large and apparently traditional school in its values, and where the way of teaching seemed to be at least mainly 'chalk-and-talk'. Whilst it is quite likely that the hypothesis regarding the

influence of the school context applies in all schools, the character of any one school will be a limitation on the research.

The method of classroom observation is of high inference judgements by the researcher, and not easily open to corroboration by, for example, another researcher. The system of recording, taking tally marks every three seconds, and observing ten categories of activities in the lesson, encourages a uniformity, and an objectivity in the observations. However, this can be seen as a limitation of the research.

In the second-stage study, similarly, the nature of the group of students chosen, as well as of the institution and the tutors of those students, just as of any group that one might choose for this research, were a factor to be noted as a limitation on the study.

9.3 Implications for Further Research

The implications for further research will be discussed in two sections: in section 9.3.1 possible extensions of the study in this thesis, and in Section 9.3.2 more general implications.

9.3.1 Extensions of Present Study

The apparent uniformity of teaching methods found in the first-stage study, despite the variation in teachers' backgrounds, training, experience and views as revealed by the questionnaire, led to the hypothesis that other factors that may be called the specific school context, are the major determinants in teachers' behaviour. If this is so, then the influence of the ethos of the school and the mathematics department could be examined further in at least three ways:

- (1) The first-stage study, based on the same structure and methodology, could be extended to take in several schools with different programmes for the mathematics curriculum. Some light would then be thrown on the influence of school structure and approach on mathematics departments and on teachers of mathematics.

(ii) A more full sociological study could be made of a single school, with the intention of revealing more information on the various influences bearing on the teaching of mathematics.

(iii) A third direction, again using the structure of the present study, would be to examine the ways that a teacher varies his or her teaching methods according to the age, ability and behaviour of the classes. There was some indication that this takes place in the data from the lesson observation in the first-stage study. More information of the lessons would be required if this were the focus of research, by for example using SCAN. It may well be, for example, that a teacher committed to an 'open' view of mathematics teaching, is more 'open' with younger classes, as they are further from public examinations, and also with better behaved classes, as a more 'closed' approach is seen as providing the teacher with more authority.

9.3.2 Other Directions

As mentioned above, the questionnaire could have uses and applications in other areas of research and in mathematics education in general. It could prove to be a useful tool in pre-service training to aid and stimulate reflection on the part of student-teachers, on the implications of their attitudes to mathematics, as has been indicated in the students' reactions in the second-stage study. It could also be used in in-service training as a means of introduction of these issues to teachers. As has been said, many teachers and of course student teachers, will be unaware of their views of the nature of mathematics, and also of the implications of those views, and a questionnaire which then places them roughly on a continuum could prove a valuable stimulus. A further possibly fruitful area could be its use as a measure of change in teachers' views over a period of time. For example, a course that made explicit the implications of attitudes to the nature of mathematics, for the teaching of mathematics, would be likely to lead to changes in the views of teachers, and the questionnaire could be a useful tool in measurement of that change.

A further important area of research that is highlighted by this thesis, is the effect on pupils' conceptions of mathematics as a result of teachers' views. How much, or how little, pupils' image of the subject changes with a new teacher holding a different view of mathematics, and/or a different teaching programme would be a valuable area of study.

An interesting spin-off from the second stage study has been the educational value of using video extracts of lessons in order to introduce students to aspects of teaching mathematics. This could certainly be examined further in pre-service teacher education. Group discussion on specific incidents and approaches could be enhanced by using such extracts.

9.4 Conclusion

This thesis may be seen as a contribution to the Theory of Mathematics Education. There is interest at the present time in the role of theory in the practice of mathematics education, and an awareness that there may be much to be gained by discussion of wider perspectives on our work. This thesis has attempted to clarify the kinds of alternatives there are for theories about mathematics, and the consequences for the teaching of mathematics, and as such attempts to provide a theoretical perspective that can be related to many areas of research in mathematical education.

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APPENDIX A

Questionnaire: First Draft and Final Draft

Marking Scheme

QUESTIONNAIRE - FIRST DRAFT

- 1) The examination syllabus largely determines the school syllabus.
- 2) The examination syllabus should not be the main factor in determining the school syllabus.
- 3) Teachers should be able to influence public examination boards in the style of examination, syllabus content etc.
- 4) Public examination syllabus, style etc. should be decided by experts not actively involved in teaching, e.g. mathematics professors, mathematicians in industry etc., but with some consultation with teachers.
- 5) Learning mathematics is essentially an active process.
- 6) Learning mathematics in school is mainly a passive process.
- 7) Creative work in mathematics only takes place at the frontiers of mathematical knowledge.
- 8) Creative work in mathematics takes place in all learning.
- 9) Expecting pupils to be creative in mathematics is unfair to pupils, and a waste of precious lesson time.
- 10) Our main aim is to try to enable every pupil to leave school with some public examination success in mathematics, GCE, CSE, RSA, or e.g. Herts Achievement.
- 11) Our main aim is to try to enable every pupil to become a mathematician to their level of ability, i.e. to be able to think mathematically where applicable.
- 12) School mathematics should be seen by pupils to be relevant and applicable to the real world.
- 13) One of our aims is to enable pupils to appreciate and enjoy mathematics for its own sake.
- 14) It is a consequence of the nature of school mathematics that pupils will wonder about the purpose of their work in mathematics as against e.g. geography.

If an intelligent pupil were to ask the purpose of a piece of work, one could answer:-

- 15) Mathematics can be seen to be, like chess, a game with rules that have to be learnt.
- 16) Mathematics is applicable to all sorts of situations. Once the knowledge is acquired, applications follow.
- 17) Mathematics is training you to be logical. Any particular piece of work is just a logical extension of previous material.
- 18) Discovery methods of learning are relevant for the earliest concepts only e.g. addition, volume etc.
- 19) Discovery methods would be useful for older pupils, if time and syllabus permitted.
- 20) School mathematics provides the foundations of mathematical knowledge, to be extended into useful and applicable skills in work or college situations.
- 21) The process of doing mathematics in school is a model of all mathematical experiences, industry, research, daily life etc.
- 22) Problem solving is part of school mathematics as it enables pupils to tackle unusual exam questions.
- 23) Problem solving is an essential part of school mathematics since it is an essential aspect of mathematical knowledge.
- 24) Problem solving is a topic of school mathematics that should be included where possible.
- 25) Asking open-ended questions is a useful device in teaching, e.g. how might we add $\frac{1}{4}$ and $\frac{1}{3}$?
- 26) Asking open-ended questions is essential to enable pupils to think for themselves, which is an essential part of school mathematics.
- 27) We never really ask open-ended questions since there is always a right way to solve the problem, and that method is known by the teacher in advance, and the pupils know that this is the case.
- 28) The word 'proof' is best defined as an explanatory process.
- 29) The word 'proof' is best defined as an explicit process of deduction.
- 30) The main work in mathematics is at the level of axioms and definitions - proofs and theorems are automatic and not as important.

- 31) The main work in mathematics is in developing proofs and finding counterexamples.
- 32) Mathematical truths are not susceptible to revolutionary change in the way that scientific truths are, e.g. relativity, quantum theory etc.
- 33) Mathematical knowledge is close to scientific knowledge in the sense that conclusions are tested for their truth.
- 34) Mathematics is essentially hierarchical. Although progress does go on making earlier work more rigorous, generally growth of knowledge builds on former work.
- 35) Mathematical truths have an inevitability about them. A world with different mathematical truths is inconceivable.
- 36) Mathematical knowledge is hypothetical and potentially subject to refutation or adaptation.

NAME :

QUESTIONNAIRE
ON
MATHEMATICS AND MATHEMATICS EDUCATION

STEPHEN LERMAN

JANUARY 1984

CTION 1

MATHEMATICS EDUCATION

- The examination syllabus largely determines the school syllabus.
- Learning mathematics in school is essentially an active process.
-) It is a consequence of the nature of mathematics itself, that pupils will more often wonder about the purpose of a topic in mathematics than in, say, geography.
-) Discovery methods of learning mathematics are relevant for the earliest concepts only, e.g. addition, volume etc.
-) School mathematics can be seen to provide the basic skills and techniques of mathematics, to be extended into applicable mathematics in work or college situations.
-) The major value of teaching problem solving skills is to enable pupils to tackle unusual examination questions.
-) The examination syllabus should not be the main factor in determining the school syllabus.
-) Learning mathematics in school is mainly a passive process.
-) Creative work in mathematics only takes place at the frontiers of mathematical knowledge.
-) Discovery methods would be useful for older pupils if time and syllabus permitted.
-) The process of doing mathematics in school can be seen to be a model of all mathematical experiences: industry, research, daily life etc.
-) Creative work in mathematics takes place in all learning.
-) Teachers should be able to influence public examination boards in the style of examination, syllabus content etc.
-) Discovery learning of mathematics is relevant for all stages of school mathematics.
- 5) The development of problem solving skills in pupils should be seen as an essential part of school mathematics.

Strongly
Agree

Agree

No
View

Dis-
agree

Strongly
Dis-
agree

	Strongly Agree	Agree	No View	Dis-agree	Strongly Dis-agree
Public examination syllabus, style etc. should be decided by experts.					
Expecting pupils to be creative in mathematics is unreasonable and doesn't warrant use of precious lesson time.					

n intelligent pupil were to ask the purpose of a c in mathematics, I would answer:					
Mathematics can be seen to be, like chess, a game with rules that have to be learnt.					
The applications of mathematics follow once mathematical knowledge is acquired.					
Mathematics is training you to be logical.					
This topic is on the syllabus, so you can rely on its importance.					

he following statements, the term 'open' is used describe questions such as "how could we add $\frac{1}{2}$ and so pupils who have not learnt the algorithm, since question could be answered in many ways.					
Asking open questions is essentially just a useful device in taeching.					
Asking open questions is vitally important as it gives pupils the opportunity for creative thought.					
There are no open questions since both teacher and pupils know that there is always only one correct way to solve any problem.					

use give an order, 1 to 4, to the following aims:					
My main aim is to try to enable every pupil to leave school with some public examination success in mathematics, GCE, CSE etc.					
My main aim is to try to enable every pupil to become a mathematician to their level of ability, i.e. to be able to think mathematically, where applicable.					
My main aim is that school mathematics should be seen by pupils (and parents, employers etc.) to be relevant and applicable to the real world.					
My main aim is to enable pupils to appreciate and enjoy mathematics for its own sake.					

PION 2

MATHEMATICS

Once a mathematical structure has been developed, and a theorem formulated, its proof is a technical detail, although it may be years till its discovery.

Mathematical truths are not susceptible to revolutionary change in the way that scientific truths are, e.g. relativity.

Mathematical knowledge is close to scientific knowledge in the sense that conclusions are tested for their truth.

Mathematical truths have an inevitability about them. A world with different mathematical truths is inconceivable.

The generation of a proof is a highly creative part of mathematics, since it can lead to new structures, reformulated hypotheses etc.

Mathematical knowledge is hypothetical and potentially subject to refutation or adaptation. Mathematics is essentially hierarchical and cumulative. Although progress does go on making earlier work more rigorous, generally new knowledge builds on former work.

Strongly Agree

Agree

No
View

Dis-
agree

Strongly
Dis-
agree

_____ This space may be used for making any comments about the questionnaire, your thoughts etc.

MARKING SCHEME

	Strongly Agree	Agree	No View	Disagree	Strongly Disagree
1)					
2)					
3)	1	2	2½	3	4
4)	1	2	2½	3	4
5)	1	2	2½	3	4
6)	1	2	2½	3	4
7)	4	3	2½	2	1
8)					
9)	1	2	2½	3	4
10)	4	3	2½	2	1
11)	4	3	2½	2	1
12)	4	3	2½	2	1
13)	4	3	2½	2	1
14)	4	3	2½	2	1
15)	4	3	2½	2	1
16)					
17)	1	2	2½	3	4
18)	1	2	2½	3	4
19)	1	2	2½	3	4
20)					
21)	1	2	2½	3	4
22)	1	2	2½	3	4
23)	4	3	2½	2	1
24)	1	2	2½	3	4
Ordering	(1)	(2)	(3)	(4)	
25)	1	2½	4	2½	
26)	4	4	2½	1	
27)	4	4	2½	1	
28)	1	1	2½	4	
	Strongly Agree	Agree	No View	Disagree	Strongly Disagree
29)	1	2	2½	3	4
30)	1	2	2½	3	4
31)	4	3	2½	2	1
32)	1	2	2½	3	4
33)	4	3	2½	2	1
34)	4	3	2½	2	1
35)	1	4	2½	3	2

APPENDIX B

Five State Project

Minnesota National Laboratories

**Categories for Interaction Analysis
of Pupil Involvement**

Instrument in use in Five State Project

Minnesota National Laboratories

CATEGORIES FOR INTERACTION ANALYSIS OF PUPIL INVOLVEMENT

TEACHER

1. Clarifying, encouraging, summarizing:
remarks which recognise student, his participation and ideas by incorporating them into classroom interaction.
2. Contacting, checking:
remarks, usually questions, which invite limited participation: asking for information, simple next steps or checking student understanding.
3. Confronting, seeking:
remarks, usually questions, which invite extensive participation: classifying information, a series of steps, a single step requiring selection and organization of material.
4. Soft or hard challenging, jolting:
remarks absurd, controversial or questions of comprehensive nature or completely undirected to invite significant participation: in noting relationships, application, in making grand leaps in system development.
5. Informing, lecturing:
remarks, expository, to provide information or solve problems; also rhetorical questions. These may be continued into a "lecture" period or be met in rapid discussion. Reading answers.
6. Directing:
remarks assigning particular tasks related to the lesson: homework assignment, a method of recording data, a type of material to use.

PUPIL

7. Receptive, passive:
remarks by student under direction limited to one-step answers, or trivial agreement or question initiated without readiness to treat it himself. Reading book or answers.

8. Independent, active

remarks by student either as invited and moving more than one step ahead, or a single step, or without invitation to raise a question and being willing to treat it himself.

9. Curious, creative:

remarks by student in which present topic related to other areas of mathematics or to applied fields, to more fundamental concepts, or to a wider family of topics. A fresh topic related to present topic.

GENERAL

0. Silence, confusion, organization:

behaviour concerned with lesson which cannot be classified 1-9.

EXAMPLES OF CLASSIFICATION OF VERBAL BEHAVIOUR IN THE TEN CATEGORIES

1. Clarifying. Encouraging. Summarizing:

- 1.1 Statements in recognition of pupil's comment.
- 1.2 Answers to pupil's question.
- 1.3 Explaining and amplifying a pupil's answer.
- 1.4 Encouragement of pupil in developing his own idea.
- 1.5 Restatement of pupil's answer.
- 1.6 Praise or comment in response to a pupil's statement, answer or question.

2. Contacting. Checking:

- 2.1 Simple passive question requiring a simple passive response.
- 2.2 Question anticipating simple one-step or trivial answers.
- 2.3 Question where range of possible answers is narrow.
- 2.4 Drill.
- 2.5 Request for information requiring simple recall or request for mere reading from text or notes.
- 2.6 Checking pupil's understanding, e.g. "Any questions", "Difficulty anyone".

3. Confronting. Seeking:

- 3.1 Statements requiring extensive participation.
- 3.2 Questions requiring independent thought.
- 3.3 A one-step answer but requiring selection and organization of material.
- 3.4 Statement eliciting pupil criticism of own work.
- 3.5 Broad questions demanding responses similar to 3.1 and 3.2 but where direction less clearly specified.

4. Soft or hard Challenging. Jolting:

- 4.1 Questions requiring very deep understanding.
- 4.2 Controversial questions of comprehensive nature.
- 4.3 Questions requiring understanding leading to relationships, applications, or grand leaps in system development.
- 4.4 Absurd statements requiring deep understanding by pupils.

- 4.5 Undirected questions inviting significant participation at a high level of understanding.
- 4.6 Statements requiring elaboration on difficult homework.
- 5. Informing. Lecturing:
 - 5.1 Expression of opinions or facts relating to mathematics.
 - 5.2 Teacher talk, but not inviting participation.
 - 5.3 Rhetorical questions and answers.
 - 5.4 Reading answers to problems (place B with it)
 - 5.5 Reading problems or information from book (place B with it)
- 6. Directing:
 - 6.1 Express statements or questions requiring delayed pupil response.
 - 6.2 Entire directions and explanations of assignments.
 - 6.3 Instructions on use of material.
- 7. Receptive. Passive:
 - 7.1 Pupil response from simple questions.
 - 7.2 Trivial statements, comments, responses.
 - 7.3 Simple question without being able to treat it himself.
 - 7.4 Drill.
 - 7.5 Reading a problem from a book (place B beside it).
 - 7.6 Reading answers to problems (place B beside it).
 - 7.7 Information of simple recall nature including answers to homework assignment.
- 8. Independent. Active:
 - 8.1 Answers to difficult questions requiring multiple-step solutions
 - 8.2 Statements made by pupil indicating powerful step or deep understanding.
 - 8.3 Responses demonstrating selection or organization of material.
 - 8.4 Suggestions of independent nature given without specific request to explain difficult points.

9. Curious. Creative:

- 9.1 High-level comments showing definite insight.
- 9.2 Pupil voluntarily relates the material to other areas of mathematics or to applied fields.
- 9.3 Pupil presents a fresh topic related to the topic under discussion.
- 9.4 An elegant solution suggested through pupil understanding (not from books).
- 9.5 Unusual application of topic.
- 9.6 Unusual generalizations.
- 9.7 Originality.
- 9.8 Humor related to subject matter.

0. Silence. Confusion. Organization:

- 0.1 The first and last entry in each daily classification to ensure totals in matrix are balanced.
- 0.2 Writing on board or writing at seats during main lesson unaccompanied by explanatory comment (place B beside it).
- 0.3 Recording of data, homework, grades, etc.

APPENDIX C

Data of First-Stage Study:

Marksheets From Questionnaire

Tally Sheets From Each Class

Teacher A

Questionnaire Results - Score 86%

	Strongly Agree	Agree	No View	Disagree	Strongly Disagree
1)		x			
2)			x		
3)		x			
4)				x	
5)			x		
6)				x	
7)				x	
8)			x		
9)				x	
10)		x			
11)		x			
12)				x	
13)	x				
14)		x			
15)		x			
16)		x			
17)				x	
18)				x	
19)		x			
20)		x			
21)				x	
22)			x		
23)		x			
24)				x	
25)2=					
26)1					
27)2=					
28)4					
29)		x			
30)				x	
31)				x	
32)			x		
33)		x			
34)		x			
35)		x			

TALLY SHEET FOR INTERACTION ANALYSIS IN THE MATHEMATICS CLASSROOM

Teacher A

Class - 4th Year Set 1

	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

TALLY SHEET FOR INTERACTION ANALYSIS IN THE MATHEMATICS CLASSROOM

Teacher A

Class - 5th Year Set 8

	0	1	2	3	4	5	6	7	8	9
0	 					 				
1						 				
2								 		
3										
4										
5			 			 				
6							8			
7		 								
8										
9										

Teacher B

Questionnaire Results - Score 77½

	Strongly Agree	Agree	No View	Disagree	Strongly Disagree
1)		x			
2)			x		
3)		x			
4)					x
5)			x		
6)			x		
7)					x
8)				x	
9)				x	
10)		x			
11)				x	
12)			x		
13)	x				
14)				x	
15)				x	
16)				x	
17)				x	
18)				x	
19)		x			
20)		x			
21)					x
22)		x			
23)		x			
24)				x	
25) 1					
26) 4					
27) 2					
28) 3					
29)			x		
30)				x	
31)		x			
32)			x		
33)		x			
34)				x	
35)			x		

TALLY SHEET FOR INTERACTION ANALYSIS IN THE MATHEMATICS CLASSROOM

Teacher B

Class - 1st Year Set R (3rd band)

	0	1	2	3	4	5	6	7	8	9
0	 					 				
1			 			 				
2								 		
3										
4										
5			 			 				
6	 									
7		 								
8										
9										

TALLY SHEET FOR INTERACTION ANALYSIS IN THE MATHEMATICS CLASSROOM

Teacher B

Class - 2nd Year Set 8

	0	1	2	3	4	5	6	7	8	9
0	 									
1			 							
2								 		
3										
4										
5			 			 				
6	 									
7		 								
8										
9										

TALLY SHEET FOR INTERACTION ANALYSIS IN THE MATHEMATICS CLASSROOM

Teacher B

Class - 1st Year Set L (1st band)

	0	1	2	3	4	5	6	7	8	9
0	 					 				
1			 			 				
2								 		
3										
4										
5	 		 			 				
6										
7		 								
8										
9										

Teacher C


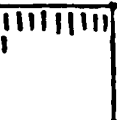





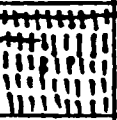









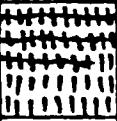

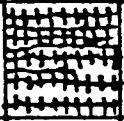







Questionnaire Results - Score 83½

	Strongly Agree	Agree	No View	Disagree	Strongly Disagree
1)		x			
2)		x			
3)		x			
4)				x	
5)				x	
6)				x	
7)				x	
8)					x
9)					x
10)			x		
11)		x			
12)				x	
13)		x			
14)		x			
15)		x			
16)			x		
17)					x
18)				x	
19)		x			
20)				x	
21)				x	
22)				x	
23)		x			
24)				x	
25)2					
26)1					
27)4					
28)3					
29)				x	
30)		x			
31)			x		
32)			x		
33)		x			
34)				x	
35)		x			

TALLY SHEET FOR INTERACTION ANALYSIS IN THE MATHEMATICS CLASSROOM

Teacher C

Class - 3rd Year Set 1

	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

Teacher D

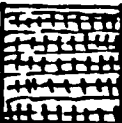


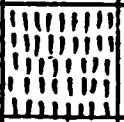
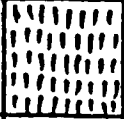

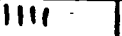

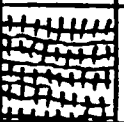





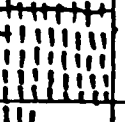

Questionnaire Results - Score 77

	Strongly Agree	Agree	No View	Disagree	Strongly Disagree
1)		x			
2)		x			
3)		x			
4)				x	
5)		x			
6)					x
7)				x	
8)				x	
9)				x	
10)				x	
11)				x	
12)				x	
13)	x				
14)			x		
15)		x			
16)				x	
17)				x	
18)				x	
19)		x			
20)		x			
21)		x			
22)				x	
23)		x			
24)					x
25)3					
26)2					
27)4					
28)1					
29)			x		
30)		x			
31)				x	
32)				x	
33)		x			
34)				x	
35)	x				

TALLY SHEET FOR INTERACTION ANALYSIS IN THE MATHEMATICS CLASSROOM

Teacher D

Class - 5th Year Set 1

	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6										
7										
8										
9										

TALLY SHEET FOR INTERACTION ANALYSIS IN THE MATHEMATICS CLASSROOM

Teacher D

Class - 2nd Year Set 4

	0	1	2	3	4	5	6	7	8	9
0	 					 				
1			 			 				
2								 		
3										
4										
5			 			 				
6	 									
7		 								
8										
9										

Teacher E

Questionnaire Results - Score 65

	Strongly Agree	Agree	No View	Disagree	Strongly Disagree
1)		x			
2)		x			
3)				x	
4)		x			
5)		x			
6)			x		
7)				x	
8)				x	
9)				x	
10)		x			
11)				x	
12)		x			
13)		x			
14)				x	
15)			x		
16)		x			
17)			x		
18)				x	
19)		x			
20)				x	
21)		x			
22)				x	
23)		x			
24)				x	
25) 1					
26) 4					
27) 2					
28) 3					
29)			x		
30)		x			
31)				x	
32)			x		
33)			x		
34)			x		
35)		x			

TALLY SHEET FOR INTERACTION ANALYSIS IN THE MATHEMATICS CLASSROOM

Teacher E

Class - 1st Year Set K (3rd band)

	0	1	2	3	4	5	6	7	8	9
0	 					 				
1						 				
2								 		
3										
4										
5			 		.	 				
6										
7		 								
8										
9										

Teacher F

Questionnaire Results - Score 82%

	Strongly Agree	Agree	No View	Disagree	Strongly Disagree
1)		x			
2)		x			
3)				x	
4)					x
5)		x			
6)					x
7)	x				
8)				x	
9)				x	
10)	x				
11)		x			
12)			x		
13)	x				
14)		x			
15)		x			
16)		x			
17)				x	
18)		x			
19)		x			
20)		x			
21)					x
22)				x	
23)	x				
24)					x
25)3					
26)1					
27)4					
28)2					
29)				x	
30)	x				
31)				x	
32)	x				
33)		x			
34)					x
35)	x				

TALLY SHEET FOR INTERACTION ANALYSIS IN THE MATHEMATICS CLASSROOM

Teacher F

Class - 3rd Year Set 3

	0	1	2	3	4	5	6	7	8	9
0	 									
1			 			 				
2								 		
3										
4										
5			 			 				
6										
7		 								
8										
9										

Teacher G

Questionnaire Results - Score 73

	Strongly Agree	Agree	No View	Disagree	Strongly Disagree
1)		x			
2)		x			
3)	x				
4)		x			
5)		x			
6)				x	
7)			x		
8)				x	
9)		x			
10)		x			
11)				x	
12)				x	
13)		x			
14)			x		
15)		x			
16)		x			
17)			x		
18)				x	
19)				x	
20)		x			
21)				x	
22)		x			
23)	x				
24)				x	
25) 1					
26) 3					
27) 4					
28) 2					
29)			x		
30)			x		
31)		x			
32)				x	
33)		x			
34)		x			
35)	x				

TALLY SHEET FOR INTERACTION ANALYSIS IN THE MATHEMATICS CLASSROOM

Teacher G

Class - 2nd Year Set 5

	0	1	2	3	4	5	6	7	8	9
0	 					 				
1			 			 				
2								 		
3										
4										
5			 			 				
6	 									
7		 								
8										
9										

Teacher H

Questionnaire Results - Score 85%

	Strongly Agree	Agree	No View	Disagree	Strongly Disagree
1)		x			
2)		x			
3)			x		
4)				x	
5)	x				
6)				x	
7)			x		
8)				x	
9)				x	
10)	x				
11)			x		
12)	x				
13)		x			
14)	x				
15)		x			
16)		x			
17)				x	
18)					x
19)		x			
20)			x		
21)		x			
22)				x	
23)	x				
24)					x
25))					
26)) no					
27)) view					
28))					
29)		x			
30)	x				
31)			x		
32)					x
33)		x			
34)				x	
35)	x				

TALLY SHEET FOR INTERACTION ANALYSIS IN THE MATHEMATICS CLASSROOM

Teacher H

Class - 2nd Year Set 1

	0	1	2	3	4	5	6	7	8	9
0	 					 				
1	"		 		 		"			
2								 		
3									"	
4										
5			 		 		"			
6	 									
7		 								
8		"								
9										

TALLY SHEET FOR INTERACTION ANALYSIS IN THE MATHEMATICS CLASSROOM

Teacher H

Class - 3rd Year Set 6

	0	1	2	3	4	5	6	7	8	9
0	 					 	 	,		
1			 			 				
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3										
4										
5			 			 				
6	 									
7		 								
8										
9										

TALLY SHEET FOR INTERACTION ANALYSIS IN THE MATHEMATICS CLASSROOM

Teacher H

Class - 5th Year Set 2

	0	1	2	3	4	5	6	7	8	9
0	 					 				
1			 			 				
2								 		
3										
4										
5			 			 				
6										
7		 								
8										
9										

Teacher I

Questionnaire Results - Score 79%

	Strongly Agree	Agree	No View	Disagree	Strongly Disagree
1)		x			
2)				x	
3)		x			
4)				x	
5)		x			
6)		x			
7)		x			
8)		x			
9)		x			
10)		x			
11)		x			
12)				x	
13)		x			
14)		x			
15)		x			
16)		x			
17)				x	
18)		x			
19)		x			
20)		x			
21)				x	
22)		x			
23)	x				
24)				x	
25)2					
26)1					
27)3					
28)4					
29)		x			
30)				x	
31)		x			
32)		x			
33)		x			
34)			x		
35)	x				

TALLY SHEET FOR INTERACTION ANALYSIS IN THE MATHEMATICS CLASSROOM

Teacher 1

Class - 2nd Year Set 2

	0	1	2	3	4	5	6	7	8	9
0	 					 				
1			 			 				
2								 		
3										
4										
5			 			 				
6	 									
7		 								
8										
9										

TALLY SHEET FOR INTERACTION ANALYSIS IN THE MATHEMATICS CLASSROOM

Teacher I

Class - 1st Year Set Q (1st band)

	0	1	2	3	4	5	6	7	8	9
0	 									
1			 							
2								 		
3										
4										
5			 			 				
6										
7		 								
8										
9										

APPENDIX D

Full Transcripts of Interviews Second-stage Study

Note: Four interviewees, A,B,C,D labelled "I" in interviews.
Researcher labelled "R" in interviews.

INTERVIEWEE A

R. O.K. Whats your first impression of the lesson?

I. One of the things that struck me was the way she had to keep looking aside, I don't know why, perhaps she had arranged her notes so that she could keep looking at the class while she was talking. And also it wasn't very clear how much they knew about simultaneous equations, so unless she had done some solving the previous lesson, I thought she should have given an example, if 3 cups of coffee are 15 pence how much is one cup? and then go on to 3 cups of coffee and 2 cups of tea, something like that. Then they would have some idea of what they were trying to do.

R. And?

I. And I think that once she had started getting them guessing for how much the tea and the coffee was - I think that was good, but maybe it went on a bit too long, I think she had too many guesses coming out and it took quite a long time. After a few guesses she should have explained what they are, what she was doing.

R. You didn't feel that they knew what they were doing?

I. I think it was clear to most of them but there seemed to be a couple there that were sitting there trying - not too sure, one was looking really bored - maybe if she had got them - I don't know how - more involved, or writing down the numbers, or maybe she could have said O.K. all do this one. If tea costs 3 pence, work out how much the coffee costs, something like that, so that they were all writing down the same answers.

R. So that they get the feel of what they are supposed to do?

I. Yes, so that at least then they would know whether they could do it or not, in that case, and if people had come up with the wrong answer, then go through it, explain how you did it and then give them another one to try and then check that everyone understood that one. But I didn't think there was much checking on whether people were understanding. I think from what we saw most people did seem to get the right answers, even though she didn't really check properly. And - I think when she put up the other, the second equation, sort of like how

much would the tea and coffee cost, it was too much of a sort of guessing game. Maybe that's just the way she introduced it - you know.

R. You felt that it was too much of a guessing game if you bring in the second equation as well?

I. Yeah. Um. Maybe if she had something like a game, three cups of coffee plus two cups of tea for 29p and three cups of coffee cost 15p. work out how much a cup of tea and a cup of coffee is, then from doing that first example on the board, 3 cups of coffee costing 15p, working out how much the coffee is, then working out the tea, and then give them both equations it might be more clear how to work out the two together.

R. What actually gives you the feeling that they weren't entirely sure what was going on?

I. Well at the beginning the long pause, when she was asking and nobody was answering and when the first few guesses came out they were sort of - (laugh) - slightly wrong? And as she was going along, although very - some of the guesses were very close that was understandable. that they could be slightly out.

R. You said that maybe that's a way of introducing simultaneous equations. How do you feel about that way?

I. I think it's good, to guess in that way - but what I've found with children at school is that if you introduce something like $x + 4 = 27$ find x . then they have no idea whatsoever, but if you say someone goes into a shop and buys some eyeshadow - or rather they have two pounds, go into a shop and buy some eyeshadow, and they have 20p change, how much did the eyeshadow cost, then they can work that out no problem. And then when you show them how to write out the equation, they seem to be able to cope with that. But when it is just letters, unless the letters mean something to them, they find it very difficult. Well that's with low ability, I don't know whether higher ability would be able to cope with it.

R. What did you feel about the ability of this group? Would you call them low ability, or what?

I. Quite low, average sort of group I'd say. Because I think some of them were brighter, getting the answer straight away. Their arithmetic was quite slow. But some of them were quite good.

R. Do you think the particular letters she used, c for coffee, t for tea, helped or hindered? Does that make it more towards the practical example you were talking about before?

I. I think, mainly, it really depends so much on what they've done, sort of beforehand, but if she'd written it all out first. like 3 cups of coffee plus 2 cups of tea = 29p or something like that, then maybe it would be clearer, and then after giving a few examples, say O.K instead of writing it all out lets see, c for coffee, t for tea and so at the same time explain that.

R. Overall then, would you say that was a good way of introducing simultaneous equations, or would you suggest some other thing that she could have done, or taught it a different way?

I. I think it was a good way, in that it was something that they all understood. They would all have bought cups of tea, cups of coffee, they would all have some understanding of what she was doing, but I think it would have been better if she had given them more hints, more guidance. It was much too open-ended.

R. Much too open-ended.

I. Well she said tell me how much tea and coffee cost, it took them much too long to try and work out what she was doing - you know maybe if she says, well suppose you gave tea some price, can anyone suggest what price tea could be? And then they said O.K. that's the price of tea, and now we've got 5 cups of tea, so how much does 5 cups of tea cost? Write that down. Now if that's how much 5 cups of tea cost, how much have we got left? So how many cups of coffee do we have to buy? And so how much would they cost? And then they would see what they were doing.

R. O.K. Any other comments, on the lesson, or the teacher's style?

I. Well, she seemed very relaxed, which I thought good. She seemed to have a very good relationship with the pupils, and they seemed very well behaved. So in that sense the teacher was good. But I think what she was teaching could have been clearer. It didn't seem as though from what we were watching, the lesson was too well thought out.

R. OK. Thank you.

INTERVIEWEE B

R. Right, what did you think of that lesson?

I. Well, at the beginning, when I saw the pictures on the wall it looked like a primary classroom, and then I saw it was a secondary classroom. Well on the blackboard you had pictures from mathematical journals, you know those pattern things. I felt that a lot of time was wasted, I mean when you observe you realize how much - well not a lot of time is wasted, but there is so little interaction going on, most of the time the teacher is waiting for someone to say something and the kids are sitting there trying to work out something - I guess that's what goes on in a lesson anyway. You have to give kids time to think, to come around to their own ideas. I felt that that way of introducing simultaneous equations is very good, to show that one equation with two variables will give you an unlimited number of solutions, and in order to have one single solution you must have a second equation. I also noticed that many of them, I think, were thinking about the price of the coffee first, because that was the first thing that came in and the tea might be something and a half, you know.

R. You said that you think it is a good way of introducing simultaneous equations. Why?

I. Well because I said it shows that with one equation with two variables you get any number of solutions which is something that, I don't know, some of the kids may not have realized, for them algebra is just algebra, they can't see the relationship between the number of variables and the number of equations you have. You might even be able to see the second of two variables, two equations. Well you start off with one and then two, and then gradually pick one out, with a third equation. They have worked their way towards finding out that there are, that you do need two equations with two variables, to find them out.

R. Did they find that out from that lesson?

I. Well, it was pointed out to them, I guess some of them got it, the ones who were participating, the ones who were finding out the various solutions to the first equation and then who realized that there was only one that fits the second equation I don't know if they realized

that there were an unlimited number of values using just one equation but they came up with different values, they could have used something and a half with coffee also something like that. I don't know it just seems as though the teacher is there, just talking away, and the kids are just writing, but only a few of them, unfortunately, most of them are probably just thinking about their own things. Not really following what's going on, not really following the thoughts that the teacher has.

R. Did you feel from the extract that you saw that that was something to do with the way the teacher was conducting the lesson, or the particular pupils or some other factor?

I. Maybe. Maybe I think it's inevitable in a lesson that something like that happens. But I know that it's probably wrong to accept this. You have to accept that you can allow a few pupils to just not be part of the lesson. Ideally you want everyone to be writing, for them to be learning, for them to be at this next stage. Well I guess they are all doing their own thing. Of course the teacher could have called them by name, asked them if they have any ideas, and they might say I don't know or the teacher might prompt them. Was this an experienced teacher?

R. I don't actually know. What do you think, what's your feeling? Judging by that extract.

I. I would say she was, I wouldn't say she was a beginner. The way she sat with them was very good. And also using teas and coffees as opposed to x's and y's was good.

R. You liked the way she used tea and coffee?

I. Oh yes, at least in terms of objects. Well there was one comment that, I mean one kid gave a suggestion that was a bit wrong because she was saying that coffee was dirt cheap and the teacher said that - you know at least it brings in a bit of judgement. I thought it was a nice way of introducing the topic. As I said, my impression throughout was that there were too many kids not participating, you only got a few talking, and they were always the same ones, the rest did nothing, they didn't know what was going on. But it's only if they actually say something that you know if they were paying attention. I won't say that they you are sure that they are following the lesson, but at least they are participating in the lesson, and what's being done on the board, as

opposed to just thinking about their own things, during the lesson, or just doodling on a piece of paper or something.

R. What do you think about the degree of direction she was giving during the lesson? Do you feel it was just about right? Or not enough, or too much?

I. I think it was about right. I didn't think it was too little, or too much or anything.

R. How did you feel about that way of introducing simultaneous equations?

I. I thought it was a good way. Yeah. As I said, it shows you the actual ideas of simultaneous equations, it, in terms of two equations. And also it's a case of interacting, answering back.

R. Supposing you were talking to that teacher, afterwards, and she said how did you feel about the way I taught?. Is there anything further you would add?

I. I think I would say that I felt that too few kids were participating. She could have prompted others who were just quiet, or she could probably see that some of them weren't even following the class, following the lesson. So that I mean she should have prompted them to attention. Sort of re-focussed their minds on the lesson as opposed to their own things. Yeah, I think that was the main thing... From the beginning I thought there was too much silence, too little interaction. And I guess the solution to that would have been that she should have asked more of the pupils to be actually involved in the lesson, by questions and answers, by prompting those that were silent.

R. How might she have done that, by calling out someone's name and say have you got an answer?

I. Well, I would have called out a name, and said what prices do you think I could put into this equation? I would ask them directly. Maybe they haven't worked out one, but I would ask them to think of one answer. But you know I would try and make them take part in it. Maybe also, when she introduced the second equation, she could have - one pupil came out with the right answer so maybe she could have asked other pupils to try and fit in the other numbers to see what the outcome is, to show them that there is only one answer that fits. Because I mean its possibly only that single pupil who came up with the right answer,

who had actually tried out any of the values. Well I mean I was watching the teacher not the pupils, and so I don't know if any of the other pupils tried answers, tried other possible values. But I guess that she could have demonstrated that the others don't fit by asking the pupils to do them. That's all really. The class were well behaved, there was no chattering. It could have been the video or something. I don't know.

R. O.K. Thank you.

INTERVIEWEE C

R. Right. What did you think of the lesson?

I. Immediately I saw, it started, a good feeling came over me, because I saw an attractive black teacher looking good, you know, in front of the classroom, my first feeling. Then that passed, and I could concentrate on what she was actually teaching. I had to assume that it was a very bright set. I didn't like the fact that she had to keep looking backwards at the book, because I think, she was teaching a mainly white class, with a few Asians, I didn't see any blacks and I think as a black teacher she has to be - she was confident, but I think she has to be even more confident in that you know, you don't - I think if you keep looking back at the book, looking back at the book, before the class, it might get a bit - you know she has to refer to the book all the time. I thought she stayed at the front - she was afraid to - you could see she was testing her limits. She was good, I liked her, I liked her, but she was afraid to move out. I gathered that she was probably fresh, the first few times, maybe she was new at the school or something.

R. What do you mean testing her limits?

I. Not willing to say right John what's the answer, or have you done it Mary, or is everybody working on it. She waited for the class, waited for the class. She got responses, maybe that's because they were a bright set. But she could have done a lot more with them. You know what I mean?

R. Like what?

I. Like walk around, make sure that everyone was doing what they should have been doing, explaining a bit more. Was that the beginning of the class? Did I see from the beginning? I did, didn't I?

R. I'm not sure, practically from the beginning, yes.

I. So she hadn't explained it. Maybe it's just my style, but I probably would have gone to - maybe after that first example I would have given the background, because I think it was a very interesting example she did, she could have done a lot with the coffee and tea very - very wide, I mean the whole class would have understood exactly what she was

talking about, there wasn't anything archaic or anything, but anyway I suppose that's just teaching styles.

R. It's that that I'm interested in, hearing your comments.

I. I'm talking about making the maths more real to the students, I mean she's talking about coffee and tea and simultaneous equations, which I never thought of - simultaneous equations in that way - I mean she could have said - right - show that those are all the different possible ways, prices you could pay for tea and coffee, we've got another equation, and do you think, how many values, you know, do you think would fit the two, or some thing. I can't think now but - you see she didn't do anything at all, you know, she just gave the equation, they threw out lots of answers, gave another equation, throw out the one answer that was correct, but I don't know how she was going to tie it in. But I would love to see the end, how she tied it in.

R. But in general you liked the example she chose?

I. Oh yes. Even as a way of explaining the relationship. If I were a student in that class, just below the average, you know, I'd have been very frustrated in a class like that. I mean all the bright ones would have given the answers. She didn't ask them for the answers, the bright ones would have just worked it out the quickest, kind of thing. And I tend to be slow. So I'm sure I would have got lost, by the time I had worked through the first one, I'd have been looking at the answer, have sat there waiting to see how she did them on the board, and how that gave 38p or whatever. And then she worked down to the next lot. So if I'm not very quick, if I'm not a quick person, I'd be very frustrated. It was obviously a very bright class. She obviously had - you could see she had a lovely personality which - a good personality for a teacher, kind of thing, so she could have done so much, I think. She didn't walk around at all, she just stayed there, she stayed, next to the blackboard. I mean she didn't really - there was no - she spoke to the class as a whole, she never spoke to any individual students. To build a relationship there. That's why I feel that she was new. She was new to the school.

R. You said something earlier about having introduced one equation and then the other one, she could have asked them something like how many

answers are there for both. Can you explain a bit more about that. Why it would have been a good thing to do.

I. Because, like I didn't - it was only when she put up the second equation that I realized the link, that it was simultaneous equations, because they had up just Relations, with the first example and all the possible solutions it didn't click then that it was simultaneous equations. When she went on to the second I said well alright she's going to link it up now, and all she did was to say if I wrote that and that equals that, which one of these will fit in? You know what she could have done, she could have said - she could have done the second equation exactly like the first, and let them throw out about 10 different solutions, and then say, right, now what do you notice? And if that goes, then say this is what we mean when we say solving equations simultaneously. There is just one. You know that kind of thing, this is what we mean by... I mean if you see it in a book and if you go away, you will see that it will make sense to you when you begin to solve equations simultaneously on your own. That's the rationale behind it then. There's just one - two different relationships between the price - you see that's why I said that this is a good example, two different relationships we wrote down between the prices of coffee and the tea, and if we want to use both simultaneously there is just one pair of prices which will satisfy it. And you have done it yourselves, you have seen that there is just one pair that satisfies it. That's the kind of thing, you know what I mean, that I was looking for, at the end, and then, something at the beginning, sort of...

R. O.K. Thank you.

INTERVIEWEE D

R. O.K. What did you think?

I. Do you mean of the teaching?

R. Yes.

I. I thought it was O.K. It was nice. She was very positive over the kids responses, she was sort of testing them but I thought they were kind of willing to put forward different - they felt able to put forward all their different answers very comfortably. It's hard to know whether the video made them behave differently, I don't know. It would me. She looked a bit like it did initially, when she - everyone was very quiet, a look came over her face, but that seemed to pass over quite quickly. I thought that it was pleasant. It was a bit sort of - in a way slow getting going, but maybe again the kids were used to being left.

R. You said she was quite receptive to their answers.

I. Yes, she took them all and she made them all right, I mean she didn't come in on the one that was right straight away, I mean the one that was really the most logical. She accepted that tea was 2 and the other was 12, she accepted that they were all right answers 'cos they fitted the equation. I mean she didn't act like at that point that she was looking for a particular one, you know she made it rather open, so that people were - felt happy making all those suggestions which I think was good.

R. What about the actual subject, as a way of introducing it?

I. I thought it was nice, I thought it was better than how I did it. It was what I had to do, but it was with younger kids. This was really nice, it was practical, it made the variables concrete, which I liked, and later, I think I would try to copy it, I thought it was a much better way of introducing it than my way, which was completely algebra, and in which I completely lost them. I thought that it was really nice, because you were estimating, guessing, and it worked very well.

R. So you liked the fact that she made the variables concrete.

I. Yes, oh yes. I think that was really good. Much less boring than teaching it straight.

R. How do you feel about how much the pupils would have appreciated what simultaneous equations were all about?

I. I don't think that they would have got the idea from just that one, but I would think with doing more they would have. I think a discussion about it, to talk about it, to draw out the fact that you are looking for the place where the two sort of equations match up, come together, sort of talk about it, and then maybe another concrete example, would help to draw it out. I don't think just one example would. But I think its a really good start.

R. You said that she made it rather open for the pupils.

I. I don't think she was too open. In some ways she was very directed. She knew what she wanted but on the other hand she did accept - she did encourage all the possible solutions. Maybe with the second equation look for possible solutions and then look to see where they overlap instead of going directly, homing in on, pick only one of these maybe they could have done the same thing with the second one, and then compare tables and see where they did overlap. And also seeing where they wouldn't overlap. I'm trying to think how you could make it more - the only other thing you could do is that the kids could make up their own combinations of tea and coffee. And see if they came up with the - I mean they could maybe use the other values to make up different combinations lots of groups of simultaneous equations. I mean you wouldn't want to confuse them too much. That might confuse them. I didn't think she was too open - she could have been more open-ended - I didn't think she was too open-ended at all.

R. You think she could have let them play around with the second equation, and that would have been more open-ended?

I. Yes.

R. And to maybe let them make up their own combinations.

I. Yes.

R. But you'd be worried about them getting confused though.

I. I would do it, but I would also be worried - I would do it and see if they get confused, I think there is a possibility. It depends how well you do it. I think if you had them working in groups and each making up another equation that was different but yet fitted in with the sets of figures so you had them talking together a bit rather than them each doing it on their own, so that they came up with maybe four pairs

of equations, or maybe five pairs, depending on the number of kids. But I liked the concrete examples much better than the way I did it.

R. How does it help?

I. It gives the kids something to grasp, understanding what you are doing, not just manipulating an expression that has unknowns - you know sometimes kids don't understand what those a's and b's are, and by giving a concrete example it gives them something to latch onto. It gives the whole exercise some meaning, it gives them something to latch onto, helps them to connect, in a way that abstract algebra - it seems to me that a lot of kids find that more difficult than any other topic, especially manipulating, something in 2 variables.

R. Anything else you want to add?

I. Just that the class seem willing and not upset by making mistakes, which seems nice, when the girl, the first person gave a wrong answer but it didn't stop everyone else from giving answers, and that was good. I mean her acceptance - she didn't say this is wrong, she tried it out and it was wrong. It made it perfectly acceptable to have a wrong answer so it made the atmosphere nice, so that someone else was willing to try. Most of the other answers were right, but it just encourages everyone else to try if you don't jump down someone's throat.

R. OK. Thank you very much.